THE HILBERT-SMITH CONJECTURE AND PRIME END THEORY ON 3-MANIFOLDS

By

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To My Parents
Chulsoo and Soonoak Lee

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The following, known as the generalized Hilbert-Smith conjecture, is the classic unsolved problem of topological transformation groups: *If G is a compact group and acts effectively on a manifold, then G is a Lie group.* In this dissertation we study various forms of the Hilbert-Smith conjecture and a new approach to its possible solution.

We give a detailed proof of the equivalence of the generalized Hilbert-Smith conjecture and the following: *A p-adic group cannot act effectively on a manifold.*

We prove that regularly almost periodic is equivalent to nearly periodic for homeomorphisms on compact metric spaces. We also prove that each of the following is equivalent to the Hilbert-Smith conjecture on compact 3-manifolds M^3: (1) If h is almost periodic on M^3, with h=identity on ∂M^3, then h=identity on M^3. (2) If h is regularly almost periodic on M^3, with h=identity on ∂M^3, then h=identity on M^3. (3) If h is regularly almost periodic on M^3, then h is periodic on M^3.

In the appendix, in a joint paper with B. Brechner we introduce a three dimensional prime end theory and an approach to the Hilbert-Smith conjecture using prime end theory on S^3.

CHAPTER 1
INTRODUCTION

The following, known as the generalized Hilbert-Smith conjecture, originated from Hilbert's 5-th problem, and is the classic unsolved problem of topological transformation groups.

CONJECTURE : *If G is a compact group which acts effectively on a manifold, then G is a Lie group.*

The conjecture is equivalent to each of the the following:

(I) *A p-adic group cannot act effectively on a manifold.*

(II) *A compact 0-dimensional infinite group cannot act effectively on a manifold.*

In Chapter 2, we will study properties of compact totally disconnected groups and will prove that each of the above two statements is equivalent to the Hilbert-Smith conjecture. I would like to thank Frank Raymond for some comments which helped in completing this chapter.

The following, known as Newman's theorem, was proved by M. H. A. Newman [N] and has also been proved by A. Dress [Dr] and P. A. Smith [S2].

Let M be a connected manifold with metric d. Then there exists an $\epsilon > 0$ such that, for every action of a finite group G on M, there exists an orbit of diameter larger than ϵ.

In Chapter 3, we will review Newman's theorem and its proof as originated from P. A. Smith [S2] and revised by G. Bredon [Bre2].

Gottschalk [G1] defined a *regularly almost periodic* homeomorphism of a metric space onto itself and proved that a regularly almost periodic homeomorphism on a

2-manifold is periodic. P. A. Smith [S3] defined a *nearly periodic* homeomorphism of a metric space onto itself and conjectured that a nearly periodic homeomorphism on a manifold is periodic. We will prove that regularly almost periodic is equivalent to nearly periodic for homeomorphisms on compact metric spaces [Theorem 4.2.1] and give an example to show that compactness is necessary.

Later we prove that each of the following statements is equivalent to the Hilbert-Smith conjecture on a compact 3-manifold M^3.

1. *If h is almost periodic on M^3, with h=identity on ∂M^3, then h=identity on M^3.*
2. *If h is regularly almost periodic on M^3, with h=identity on ∂M^3, then h=identity on M^3.*
3. *If h is regularly almost periodic on M^3, then h is periodic on M^3.*
4. *If h is nearly periodic on M^3, then h is periodic on M^3.*
5. *(Newman's property for regularly almost periodic homeomorphisms). Let h be a regularly almost periodic homeomorphism of M^3 onto itself. Then there exists $\epsilon > 0$ such that if $i \in Z_+$ with $d(x, h^{iZ}(x)) < \epsilon$ for all $x \in M^3$, then h^i acts trivially on M^3.*

Hence, if we can prove one of these conditions the Hilbert-Smith conjecture will follow.

One motivation for Chapter 4 is the following question, raised by B. Brechner in [Br2]: If h is almost periodic on B^3, with h=identity on ∂B^3, then must h be the identity on B^3? A. Fathi gave me a valuable comment in conversation for generalization of Proposition 4.3.1.

In the appendix, in a joint paper with B. Brechner we introduce a three dimensional prime end theory and use it to obtain some results about p-adic group action on E^3 and S^3.

Prime end theory is essentially a compactification theory for simply connected, bounded domains, U, in E^2, or simply connected domains in S^2 with nondegenerate complement. The planar case was originally due to Caratheodory [C], and was later generalized to the sphere by Ursell and Young [U-Y], and to arbitrary two manifolds by Mather [Mat]. For each such domain, U, there is given an associated structure of crosscuts, chains of crosscuts, prime ends, and impressions of prime ends. Caratheodory [C] and Ursell and Young [U-Y] proved the following:

Theorem 1.0.1 [C] The prime ends of U are in 1-1 correspondence with the boundary points of the unit disk. That is, the compactification is by a manifold.

Theorem 1.0.2 [U-Y] There is a C-transformation $\phi : U \to Int(D)$ such that ϕ is uniformly continuous on the collection of crosscuts of U, although not necessarily on U.

*Theorem 1.0.3 [U-Y] (*The Induced Homeomorphism Theorem*) Let U be a simply connected domain in the plane, and let $h : Cl(U) \to Cl(U)$ be a homeomorphism. Let $\phi : U \to Int(D)$ be a C-transformation. Then $\phi h \phi^{-1} : Int(D) \to Int(D)$ can be extended to a homeomorphism of D onto itself.*

There are many applications of the two dimensional theory, including applications to fixed point problems, embedding problems, periodic points of homeomorphisms, and homeomorphism (group) action and extension problems. See, for example, [C-L, Mas, Ep2, Br2,3,4, Br-Mau, Br-May, May1,2, Lew, and Mat], among others.

Several constructions of a three dimensional theory appear in the literature, including work by Kaufmann [Kau], Mazurkiewicz [Maz], and Epstein [Ep1]. These papers have not yet had any applications of which we are aware.

In this paper, we develop a *simple* three dimensional prime end theory for certain open subsets of Euclidean three space. It includes conditions not addressed by any of the above three authors. Our additional conditions focus on an "Induced Homeomorphism Theorem," which we believe provides the necessary ingredient for applications. In particular, we obtain some theorems with potential applications to the Hilbert-Smith Conjecture.

We wish to acknowledge receipt of McAuley's preprint [Mc], which asserts that the Hilbert-Smith conjecture is true.

CHAPTER 2
THE HILBERT-SMITH CONJECTURE

2.1 Construction of the P-adic Transformation Group

In this section we will construct the p-adic transformation group using different methods and study its properties. Basically this section comes from [M-Z].

Let M be a Hausdorff space and G a topological group, each element of which is a homeomorphism of M onto itself. Let

$$\Phi : G \times M \longrightarrow M$$

be the map defined by

$$\Phi(g, x) = g(x) = x' \in M, \text{ for } g \in G \text{ and } x \in M.$$

We say the pair (G, M) or sometimes, G itself, is a *topological transformation group* iff, for every pair of elements $g_1, g_2 \in G$ and every $x \in M$,

(1) $\Phi((g_1 g_2), x)) = g_1(g_2(x))$ and

(2) $\Phi : G \times M \longrightarrow M$ is continuous.

From (1) and the fact that each $g \in G$ is one to one on M, it follows that

(3) $e(x) = x$; e is the identity in G.

If e is the only element in G which leaves all of M fixed, then the action of G is called *effective.*

A topological group is called *locally compact* iff the group space is locally compact. Since group translations are homeomorphisms, a local property holds at every point if it holds at some one point, say the identity. Therefore a topological group is locally compact iff the identity has a compact neighborhood.

Now let p be a prime number and let D_p be the set of all formal series in powers of p :

$$g = a_0 + a_1 p + ... + a_n p^n + ..., \text{ each } a_n = 0, ..., p-1.$$

If we add elements with infinite carry-over, then D_p forms an Abelian group and the topology is determined by the following choice of neighborhoods of the identity:

$$U_m = \{g \in D_p | a_i = 0 \text{ if } i < m\}, m = 1, 2, ...$$

We call D_p the *p-adic group*, and if D_p acts on a Hausdorff space M onto itself, as a group of homeomorphisms, we say that D_p is the *p-adic transformation group* acting on M.

A Cantor set can be defined as follows: Let

$$A_i = \{0, 1, ...p-1\} \text{ with discrete topology.}$$

If we put the product topology on $\prod_{i=1}^{\infty} A_i$, then $\prod_{i=1}^{\infty} A_i$ is homeomorphic to the Cantor set which is constructed by a geometric method [Du, pg.104] and it is totally disconnected and compact.

We define a map

$$\phi : \prod_{i=1}^{\infty} A_i \longrightarrow D_p, \text{ by } \phi(a_0, ..., a_n, ...) = (a_0 + ... + a_n p^n + ...).$$

Then the map ϕ is continuous and one to one since

$$\phi^{-1}(U_n) = \{< 0, 0, ..., 0, a_n, a_{n+1}, ... >\} \text{ for } U_n \subset D_p$$

and the homeomorphism follows from the fact that D_p is a Hausdorff space and $\prod_{i=1}^{\infty} A_i$ is compact.

Another important construction of the p-adic group, which we will use later, is the following:

Let D_p be the p-adic group which we already constructed. Then

$$U_m = \{g \in D_p | a_i = 0 \text{ if } i < m\}, m = 1, 2, ...$$

form open subgroups and hence closed subgroups, since the cosets of U_m are open in D_p. We consider the sequence of quotient groups

$$D_p/U_0, D_p/U_1, ...D_p/U_n, ...$$

For $j > i$, let

$$h_{i,j} : D_p/U_j \longrightarrow D_p/U_i$$

be the homomorphisms defined by $gU_j \longrightarrow gU_i$. Then we have

$$D_p \simeq \lim_{\leftarrow}\{D_p/U_j\} \text{ with bonding map } h_{i,j}.$$

We notice that D_p/U_i is a cyclic group of order p^i. Therefore we can also define the p-adic group as the inverse limit of cyclic groups of order p^i for $i = 1, 2, \ldots$.

2.2 Totally Disconnected Transformation Groups

Let G be a locally compact group (e.g. p-adic group) and H an open subgroup of G. Then, since the cosets of H are open in G, H is closed and G/H is discrete.

Now let U be an open and compact neighborhood of e. Then U contains a compact open subgroup. In particular, if G is a totally disconnected locally compact group and U is a compact neighborhood of e, then U contains a compact open subgroup. [M-Z, pg. 54]

Let K be a compact subset of a locally compact group G, and let U be an open neighborhood of e. Then there exists a neighborhood V of e such that $x^{-1}Vx \subset U$ for $x \in K$ and therefore $V \subset xUx^{-1}$ for $x \in K$. [M-Z, pg. 55]

We notice that we can also have $xVx^{-1} \subset U$ for $x \in K$ from the proof of the above statement. Since V does not depend on $x \in K$,

$$\cap_{x\in K} x^{-1}Ux$$

is a neighborhood of e.

The following theorem plays an important role in the study of the Hilbert-Smith conjecture. It is from the following theorem that a totally disconnected compact group can be approximated by finite groups; i.e., it is the inverse limit of finite groups.

<u>Theorem 2.2.1</u> *[M-Z,pg. 56]. If G is a totally disconnected compact group and a neighborhood U of e is given, then there is a compact normal open subgroup $H \subset U$ such that G/H is a finite group.*

Proof. From the above, U must contain a compact open subgroup, call it H'. Set $H = \cap x^{-1}H'x$, for all $x \in G$. Let V be a neighborhood of e such that $V \subset H$. For every $a \in H$, $a = ae \in aV \subset aH = H$. Therefore the group H is open and compact by the above and is normal, since

$$y^{-1}(\cap x^{-1}H'x)y \subset \cap(xy)^{-1}H'(xy).$$

Since G/H is compact and discrete, G/H must be finite. □

We now turn to the discussion of inverse limits. Assume $A_i (i \in I)$ is a system of groups, indexed by a directed set I, and for each pair $i, j \in I$ with $i \leq j$ there is given a homomorphism

$$\pi_i^j : A_j \rightarrow A_i (i \leq j)$$

such that

(i) π_i^i is the identity map of A_i, for each $i \in I$,

(ii) for all $i \leq j \leq k$ in I, we have $\pi_i^j \pi_j^k = \pi_i^k$.

Then the system $A = \{A_i (i \in I); \pi_i^j\}$ is called an *inverse system.* The *inverse limit* of this system,

$$A^* = \lim_{\leftarrow} A_i,$$

is defined to consist of all vectors $a = (..., a_i, ...)$ in the direct product $A = \prod A_i$ for which $\pi_i^j(a_j) = a_i (i \leq j)$ holds.

From the following theorem, we can characterize the inverse limit group.

<u>Theorem 2.2.2</u> *[Fu, pg. 61]. The inverse limit A^* of the inverse system $A = \{A_i (i \in I); \pi_i^j\}$ has this property: if G is a group and if there are homomorphisms $\sigma_i : G \rightarrow A_i$ with commutative diagrams,*

$$\begin{tikzcd} G \arrow[d, "\sigma_j"'] \arrow[dr, "\sigma_i"] & \\ A_j \arrow[r, "\pi_i^j"] & A_i \end{tikzcd} \qquad (i \le j),$$

then there exists a unique homomorphism $\sigma : G \to A^*$:

$$\begin{tikzcd} G \arrow[r, "\sigma"] \arrow[dr, "\sigma_i"'] & A^* \arrow[d, "\pi_i"] \\ & A_i \end{tikzcd} \qquad (i \in I)$$

for which all the diagrams are commutative, where π_i is the canonical homomorphism. This property characterizes A^* and π_i up to isomorphism. □

<u>*Theorem 2.2.3*</u> *[Fu, pg. 62]. Let*

$$A = \prod_{i \in I} B_i$$

be the direct product of groups B_i. Define J to consist of all finite subsets α of the index set I where $\alpha \le \beta$ means "α is a subset of β." For $\alpha \in J$, let $A_\alpha = \bigoplus_{i \in \alpha} B_i$ and for $\alpha \le \beta$ let π_α^β denote the projection map $A_\beta \to A_\alpha$. Then

$$A^* = \varprojlim A_\alpha \cong A.$$

Proof. Let π_α denote the canonical maps in the theorem and σ_α the projections $A \to A_\alpha$. By the Theorem 2.2.2, there is a unique $\sigma : A \to A^*$ such that $\pi_\alpha \sigma = \sigma_\alpha$. If $\sigma a = 0$ for some $a \in A$, then $\sigma_\alpha a = \pi_\alpha \sigma a = 0$ for all $\alpha \in J$, and so $a = 0$ and therefore σ is one to one.

To show that σ is onto, let $a^* = (..., a_\alpha, ..., a_\beta, ...) \in A^*$. Then we write $a_\alpha = b_{i_1} + ... + b_{i_k} (b_i \in B_i)$, where $\alpha = \{i_1, ..., i_k\}$. Because of the choice of π_α^β, $i_1 \in \beta$ and this implies the i_1-th coordinate of a_β must be b_{i_1}; hence a^* defines a unique

$(..., b_i, ...) \in A$. From the definition of σ we have $\sigma(..., b_i, ...) = a^*$, so σ is onto, and hence an isomorphism. □

Corollary 2.2.1 Let P be the set of all prime numbers and let

$$A = \prod_{p \in P} Z_p.$$

Let G_i be the finite group such that $G_i \cong Z_2 \oplus Z_3 \oplus ... \oplus Z_{p_i}$ where p_i is the i-th prime number and let $n_i = \{2, 3, ..., p_i\}$. Then

$$\lim_{\leftarrow} G_i \cong A.$$

Proof. Let J be the set of all finite subsets α of index set P. Then, from the Theorem 2.2.3, we have

$$A \cong \lim_{\leftarrow J} A_\alpha$$

where A_α is the direct sum of cyclic groups with prime order which appears in α. We notice that the set $\{n_i : i \in N\}$,where $n_i = \{2, 3, ..., p_i\}$, is a cofinal system of J, and therefore $\lim_{\leftarrow} G_i \cong A$. □

Theorem 2.2.4 Let H be a compact totally disconnected infinite group acting effectively on a manifold. Then H must contain a p-adic group, for some prime p.

Before proving the theorem, we look at the following example.

Example. Let $\{G_i\}$ be direct sums of finite cyclic groups defined by

$$G_i \cong Z_2 \bigoplus Z_3 \bigoplus ... \bigoplus Z_{p_i}.$$

Then the inverse limit group G is isomorphic to $\prod_{p \in P} Z_p$ by the Corollary 2.2.1. We show that the group G can not act effectively on a manifold. In fact, clearly the group G does not contain any p-adic group. Now we consider the small subgroups. Then the small subgroups have the form $\prod_{p \geq p_k} Z_p$. Since direct sums are subgroups,

the small subgroups contain elements of finite order. This contradicts Newman's theorem [N]. Consequently the group G can not act effectively on a manifold.

Proof of Theorem 2.2.4. Let H be a compact totally disconnected infinite group acting effectively on a manifold. Note that H must contain an element $g \in H$ with infinite order [Ya2]. We consider a sequence of compact invariant subgroups, $H_1 \supset H_2 \supset H_3 \supset \ldots$ with $\lim H_i = e$ and $\lim_{\leftarrow} H/H_i \cong H$. Let g_i denote the coset gH_i in H/H_i, and let A_i be the finite subgroup of H/H_i which is homomorphic image of the cyclic group S, generated by g. Then A_i is cyclic, since A_i is homomorphic image of S. Consider $\{A_i\}$ and the onto homomorphism $\pi_i^j|_{A_j}$, where π_i^j is the natural projection from $H/H_j \to H/H_i$. Now we take the inverse limit of $\{A_i\}$ with bonding maps $\pi_i^j|_{A_j}$, denoted by A^*. Then, $A^* \cong \overline{S}$ [S3] and by the following theorem, Theorem 2.2.5, $A^* \cong \prod_{p_m} Z_{p_m^{e_m}}$, where p_m is a prime number which appears in the product of prime powers of the order of A_i for some i. We note that $e_m = \infty$ means that $Z_{p_m^{e_m}}$ is a p_m-adic subgroup.

Now we suppose that the group H does not contain any p-adic subgroup. Then e_m is finite for each m.

We apply the above argument to a compact invariant subgroup H_i for all i. Therefore, for every H_i, H_i contains a subgroup which is isomorphic to $\prod_{p_k} Z_{p_k^{e_k}}$ where e_k is finite for each k, since the group H does not contain any p-adic subgroup.

Consequently, given any $\epsilon > 0$ there exists an element h_ϵ such that the orbit of h_ϵ has diameter less than ϵ, and the order of h_ϵ is finite. This contradicts Newman's theorem [N] and completes the proof. □

<u>Theorem 2.2.5</u> Let $\{A_i\}$ be a sequence of finite cyclic groups such that for $j > i$, the order of A_j is a multiple of the order of A_i. Let $\pi_i^j : A_j \to A_i$ be an onto homomorphism. Then $\lim_{\leftarrow} A_i \cong \prod_{p_m} Z_{p_m^{e_m}}$ where p_m is a prime number which appears in the product of prime powers of the order of A_i for some i and p_m's are all different prime numbers.

Proof. Let n_i be the order of A_i with $n_i = p_1^{a_{1i}} p_2^{a_{2i}} ... p_k^{a_{ki}}$. Then

$$A_i \cong Z_{p_1^{a_{1i}}} \bigoplus ... \bigoplus Z_{p_k^{a_{ki}}}$$

and for $j > i$, the order of A_j, $n_j = p_1^{a_{1j}} p_2^{a_{2j}} ... p_k^{a_{kj}} ... p_l^{a_{lj}}$ with $p_m^{a_{mi}} \mid p_m^{a_{mj}}$ for $m = 1, 2, ..., k$.

Now we consider the inverse limit of each column corresponding to a prime p_m where p_m is a prime number which appears in the product of prime powers of the order of A_i for some i. [Notation: We call a sequence of subgroups of $A_i's$, where the sequence is isomorphic to $\{Z_{p_m^{e_{m1}}}, Z_{p_m^{e_{m2}}}, ..., Z_{p_m^{e_{mi}}} ...\}$, the column corresponding to a prime p_m. See the example in Figure 2.2.1]

$$\begin{array}{ccccccccccc}
A_1 & \cong & Z_2 & & & & & & & & \\
\uparrow & & \uparrow & & & & & & & & \\
A_2 & \cong & Z_4 & + & Z_3 & & & & & & \\
\uparrow & & \uparrow & & \uparrow & & & & & & \\
A_3 & \cong & Z_{16} & + & Z_9 & + & Z_7 & + & Z_{11} & & \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
A_4 & \cong & Z_{16} & + & Z_9 & + & Z_{49} & + & Z_{11} & + & Z_{19} \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

FIGURE 2. 2. 1

Then the inverse limit of each column corresponding to a prime p_m is isomorphic to $Z_{p_m{}^{e_m}}$ where $e_m = \infty$ means that $Z_{p_m{}^{e_m}}$ is a p_m-adic subgroup with bonding map π_{mi}^{j} from the cyclic subgroup of A_j with order $p_m^{e_{mj}}$ to the cyclic subgroup of A_i with order $p_m^{e_{mi}}$ induced by π_i^j,which is a bonding map from A_j to A_i .

Now we show that

$$A^* = \varprojlim A_i \cong \prod_{p_m} Z_{p_m^{e_m}}$$

We define a map $\sigma_i : \prod_{p_m} Z_{p_m^{e_m}} \rightarrow A_i$ by

$\sigma_i(a_1, a_2, ...) = (\pi_{1i}(a_1), \pi_{2i}(a_2), ..., \pi_{ki}(a_k))$ where π_{mi} is the canonical projection from the inverse limit of the column corresponding to a prime p_m, $Z_{p_m^{e_m}}$, induced by

bonding maps π^j_{mi} to that subgroup of A_i, which is isomorphic to $Z_{p_m^{e_{mi}}}$; i.e., σ_i is the composition of the quotient map from $\prod_{p_m} Z_{p_m^{e_m}}$ to $Z_{p_1^{e_1}} \oplus \ldots \oplus Z_{p_k^{e_k}}$ and the map $(\pi_{1i}, \pi_{2i}, \ldots, \pi_{ki}) : Z_{p_1^{e_1}} \oplus \ldots \oplus Z_{p_k^{e_k}} \to A_i$.

We now show that the diagram of Theorem 2.2.2 commutes.

$$\begin{tikzcd} \prod Z_{p_m^{e_m}} \arrow[d, "\sigma_j"'] \arrow[dr, "\sigma_i"] & \\ A_j \arrow[r, "\pi_i^j"] & A_i \end{tikzcd} \qquad (i \leq j),$$

Let $(a_1, a_2, \ldots) \in \prod_{p_m} Z_{p_m^{e_m}}$. Then $\sigma_i((a_1, a_2, \ldots)) = (\pi_{1i}(a_1), \pi_{2i}(a_2), \ldots, \pi_{ki}(a_k))$ and

$\sigma_j((a_1, a_2, \ldots) = (\pi_{1j}(a_1), \pi_{2j}(a_2), \ldots, \pi_{kj}(a_k), \ldots, \pi_{lj}(a_l))$. Then by the map $\pi_i^j : A_j \to A_i$, $\pi_i^j(\pi_{mj}(a_m)) = \pi_{mi}(a_m)$ for $m = 1, 2, \ldots, k$. So the diagram commutes.

Therefore there exists a unique homomorphism $\sigma : \prod_{p_m} Z_{p_m^{e_m}} \to A^*$ such that the following diagram commutes, by Theorem 2.2.2 where $\sigma g = (\sigma_1(g), \sigma_2(g), \ldots)$.

$$\begin{tikzcd} \prod Z_{p_m^{e_m}} \arrow[r, "\sigma"] \arrow[dr, "\sigma_i"'] & A^* \arrow[d, "\pi_i"] \\ & A_i \end{tikzcd} \qquad (i \in N)$$

We show that σ is one to one. Let $a = (a_1, a_2, \ldots) \in Ker\,\sigma$. Then, by the commutativity of the diagram, we have $\sigma_i a = \pi_i \sigma a = 0$ for all i. Note that $\sigma_i a = (\pi_{1i}(a_1), \pi_{2i}(a_2), \ldots, \pi_{ki}(a_k)) = 0$. Therefore for all i and for all m, $\pi_{mi}(a_m) = 0$, hence $a = 0$.

Now we show that σ is onto. Let $(b_1, b_2, \ldots) \in A^*$ where $b_i \in A_i$, $b_i = b_{1i} \oplus b_{2i} \oplus \ldots \oplus b_{mi}$. We define $b_{m*} = (b_{m1}, b_{m2}, b_{m3}, \ldots)$ where b_{mj} is the element of the column corresponding to p_m in A_j. Then b_{m*} is the element of the inverse limit induced by p_m column of $\{A_i\}$, which is isomorphic to $Z_{p_m{}^{e_m}}$. Then $(b_{1*}, b_{2*}, \ldots) \in \prod_{p_m} Z_{p_m^{e_m}}$. We

claim that $\sigma(b_{1*}, b_{2*}, \ldots) = (b_1, b_2, \ldots)$. Note that $\sigma_i(b_{1*}, b_{2*}, \ldots) = (\pi_{1i}(b_{1*}), \pi_{2i}(b_{2*}), \ldots, \pi_{ki}(b_{k*})) = (b_{1i}, b_{2i}, \ldots, b_{ki}) = b_i$. Recall that $\sigma g = (\sigma_1(g), \sigma_2(g), \ldots)$. Therefore $\sigma(b_{1*}, b_{2*}, \ldots) = (\sigma_1(b_{1*}, b_{2*}, \ldots), \sigma_2(b_{1*}, b_{2*}, \ldots), \ldots) = (b_1, b_2, \ldots)$; hence, σ is onto and this completes the proof. □

2.3 The Hilbert-Smith Conjecture

The following statement, known as the generalized Hilbert-Smith conjecture, remains unresolved.

CONJECTURE : *If G is a compact group and acts effectively on a manifold, then G is a Lie group.*

The conjecture is equivalent to each of the following statements:

(I) *A p-adic group cannot act effectively on a manifold.*

(II) *A compact 0-dimensional infinite group cannot act effectively on a manifold.*

In this section we will prove that the above three statements are equivalent. Essentially the equivalence comes from the following:

Theorem 2.3.1 *If G is a compact non-Lie group acting effectively on a manifold, then G contains a p-adic group for some prime p.*

Note: This theorem was stated by Raymond [R1] without proof.

For the proof of the theorem we will use the following well known facts.

Proposition 2.3.1 *[Gl, Y1. See also M-Z, pg. 107]. If G is a locally compact group without small subgroups, then G is a Lie group.*□

The following proposition, known as the Structural Theorem for Locally Compact Groups, was proved by Yamabe [Y2].

Proposition 2.3.2 *[Y2. See also M-Z, pg. 175]. If G is a compact non-Lie group, then there exists a sequence of compact invariant subgroups $H_1 \supset H_2 \supset \ldots$ such that* $\lim H_i = e$, *giving Lie factor groups and G is the inverse limit of* $\{G/H_i\}$. □

Proposition 2.3.3 *[M-Z, pg. 237]. Every n-dimensional locally compact group G, in some neighborhood U of the identity, is the direct product of a compact totally disconnected group H and a local n-parameter Lie group R;*

$$U = H \times R.$$

We may suppose that R is ruled by one-parameter local subgroups. If K is a compact invariant subgroup of G such that G/K is a Lie group, then $\dim G/K \leq n$. □

Proposition 2.3.4 *[Ya2] If G is a compact group acting effectively on a manifold and if every element of G is of finite order, then G is a finite group.*

Proof. Since every element of G is of finite order, the powers of each element of G can not be uniformly small by Newman's theorem; i. e. G can not have arbitrarily small subgroups. Therefore G is a compact Lie group, by Proposition 2.3.1.
Now to show that G is finite, it suffices to show that G has no nontrivial one-parameter subgroups, by Lemma 2.3.1 below. But this is clear since if G has a nontrivial one-parameter subgroup then G has an element with infinite order. □

Lemma 2.3.1 *[Gl]. Let G be a locally compact group which is not discrete and does not have arbitrarily small subgroups. Then G has a nontrivial one-parameter subgroup.* □

Corollary 2.3.1 *Let M be a compact manifold such that every nearly periodic transformation which acts on M is periodic. Then any compact totally disconnected group G acting on M is finite. That is, a compact totally disconnected infinite group can not act effectively on M.* □

Proof. Let $g \in G$ then g is nearly periodic [S3]. □

Remark: The statement in Proof was mentioned by P. A. Smith [S3] without proof. This is proved in Proposition 4.2.1 in Chapter 4.

Proof of the Theorem 2.3.1. Let G be a compact non-Lie group. Then G contains a compact totally disconnected infinite subgroup H, by Proposition 2.3.3. Since G acts effectively, H also acts effectively. Therefore H must contain a p-adic group for some prime p, by Theorem 2.2.4. □

Now we can show that the generalized Hilbert-Smith conjecture is equivalent to the statements (I) and (II).

The generalized Hilbert-Smith conjecture implies (I): Let A_p be a p-adic group acting on manifold M. Then, since A_p is a non-Lie group, A_p can not act effectively on M. Conversely, suppose there exists a compact group G acting effectively on M which is a non-Lie group. Then G contains p-adic group for some prime p, by Theorem 2.3.1. (I) implies (II): Suppose that there exists a compact 0-dimensional infinite group G acting effectively on M. Then, also by Theorem 2.3.1, G contains a p-adic group for some prime p. The converse is clear.

We now prove an additional theorem about the structure of compact non-Lie groups.

<u>Theorem 2.3.2</u> *If G is a compact non-Lie group acting effectively on a manifold M, then G contains a compact totally disconnected Abelian non-Lie group.*

Proof. Suppose that G is a compact non-Lie group acting effectively on a manifold M. Then G is not finite; therefore G contains an element g of infinite order by Proposition 2.3.4. Now we consider the closed subgroup $\overline{H}$ of the cyclic subgroup generated by g. Then $\overline{H}$ is an Abelian subgroup and compact since G is compact.

Now we show that G contains a compact totally disconnected Abelian non-Lie group. Notice that there exists a sequence of compact invariant subgroups of $\overline{H}$, $Z_1 \supset Z_2 \supset \ldots$, such that $\lim Z_i = e$, by Proposition 2.3.2. We now apply the above argument to each Z_i. Then we have a sequence of compact Abelian subgroups $\{\overline{H_i}\}$. We claim that there exists a non-Lie group $\overline{H_i}$ in $\{\overline{H_i}\}$. Since if every $\overline{H_i}$ is a Lie group,

then that contradicts Newman's Theorem for compact Lie groups. Consequently, if G is a compact non-Lie group acting effectively on a manifold, then G must contain a compact Abelian non-Lie group Z. Now, if we apply Proposition 2.3.3 to a compact Abelian non-Lie group Z, then Z contains a compact totally disconnected Abelian non-Lie group. This completes the proof. $\square$

CHAPTER 3
NEWMAN'S THEOREM

3.1 Special Homology Groups

Let G be a finite group and K a simplicial complex. We say that G acts on K *simplicially* iff each transformation $g \in G$ is a simplicial map. The simplicial complex K, together with such an action G, is called a *simplicial G-complex.*
A simplicial action of G on K is called *regular* iff the action of each subgroup H of G satisfies the following [Bre2]:

If $g_0, g_1, ..., g_n$ are elements of H and $(v_0, v_1, ..., v_n)$ and $(g_0v_0, g_1v_1, ..., g_nv_n)$ are both simplices of K then there exists an element $g \in H$ such that $g(v_i) = g_i(v_i)$ for all i.

In general, a simplicial action of G on K is not a regular action. For example, the cyclic permutation of three vertices of a 2-simplex is not a regular action. But, on the second barycentric subdivision of a complex K, any simplicial action becomes regular. Therefore an assumption of regularity is no loss of generality from the topological viewpoint [Bre2].

Let K^G be the subcomplex of a regular G-complex K consisting of all simplices which are pointwise fixed under G. Notice that, for any $g \in G$ and simplex s of K, $s \cap g(s)$ is pointwise fixed.

In fact, if v, gv belong to same simplex, then (v, v) and (v, gv) are simplices of K. There exists g' such that $v = g'v$ and $gv = g'v$ by definition of regular G-complex. Therefore $gv = g'v = v$. Now let s be a simplex, $g \in G$ and $v \in s \cap gs$. Then $gv \in gs \cap g^2s$, so $(v, gv) \in gs$. Consequently $gv = v$, and hence $s \cap g(s)$ is pointwise

fixed.

So, if $x \in |K|^G$ with $x = \sum_{i=0}^{n} \alpha_i v_i$, then, for any $g \in G$,

$$g(v_0, v_1, ..., v_n) \cap (v_0, v_1, ..., v_n) \neq \emptyset$$

and is pointwise fixed by the regularity of G. Therefore $(v_0, v_1, ..., v_n) \subset K^G$, and hence $x \in |K^G|$. So we have

$$|K^G| = |K|^G.$$

Now we shall restrict our attention to a multiplicative group G of prime order p and shall study homology (cohomology) with coefficients in Z_p.

Let K be a regular G-complex and let g be a generator of G and put

$$\sigma = 1 + g + ... + g^{p-1}$$

$$\tau = 1 - g$$

in the group ring Z_pG. Since $g^p = 1$, we have $\sigma\tau = 0 = \tau\sigma$. Since we are working over Z_p, we have that $\sigma = \tau^{p-1}$. If $\rho = \tau^i$, we put $\overline{\rho} = \tau^{p-i}$. Thus $\tau = \overline{\sigma}$ and $\sigma = \overline{\tau}$.

Let $L \subset K$ be an invariant subcomplex. We consider the chain subcomplex

$$\rho C(K, L; Z_p)$$

of $C(K, L; Z_p)$ for $\rho = \tau^i, 1 \leq i \leq p-1$. (This is possible since $g\partial = \partial g$ where $g \in G$ and ∂ is the chain map.)

The basic result is the following theorem.

<u>Theorem 3.1.1</u> [Bre2, p. 122]. For each $\rho = \tau^j, 1 \leq j \leq p-1$,

$$0 \to \overline{\rho}C(K, L; Z_p) \oplus C(K^G, L^G; Z_p) \xrightarrow{i} C(K, L; Z_p) \xrightarrow{\rho} \rho C(K, L; Z_p) \to 0$$

is an exact sequence of chain complexes, where i is the sum of the inclusions and $\rho : c \mapsto \rho c$. □

For $\rho = \tau^i, 1 \leq i \leq p-1$, we define

$$H^\rho(K, L; Z_p) = H(\rho C(K, L; Z_p)).$$

This graded group is called the *Smith special homology group*.

From standard facts (e.g. Zig Zag Lemma), the short exact sequence of the theorem gives a long exact sequence in homology;

$$\ldots H^\rho_{n+1}(K, L; Z_p) \xrightarrow{\delta_*} H^{\bar{\rho}}_n(K, L; Z_p) \oplus H_n(K^G, L^G; Z_p) \xrightarrow{i_*} H_n(K, L; Z_p) \xrightarrow{\rho_*} \ldots .$$

Now we dualize the above results to cohomology. Thus one defines the *Smith special cohomology group*

$$H_\varrho{}^*(K, L; Z_p) = H^*(\varrho C(K, L; Z_p)),$$

that is, the homology of the cochain complex $Hom(\varrho C(K, L; Z_p), Z_p)$. The long exact sequence of homology, which we constructed the above, dualizes to an exact sequence of cohomology groups;

$$\ldots H^{n+1}_\rho(K, L; Z_p) \xleftarrow{\delta^*} H^n_{\bar{\rho}}(K, L; Z_p) \oplus H^n(K^G, L^G; Z_p) \xleftarrow{i^*} H^n(K, L; Z_p) \xleftarrow{\rho^*} \ldots .$$

3.2 G-coverings and Čech theory.

Let G be a finite group and X a space. If $\mathcal{U}$ is an open covering of X and if $g \in G$, then $g(\mathcal{U}) = \{g(U) : U \in \mathcal{U}\}$ is also an open covering. If $g(\mathcal{U}) = \mathcal{U}$, for all $g \in G$, we say that $\mathcal{U}$ is an *invariant* covering.

If $\mathcal{U}$, $\mathcal{V}$ are coverings, then

$$\mathcal{U} \cap \mathcal{V} = \{U \cap V : U \in \mathcal{U}, \mathcal{V} \in \mathcal{V}\}$$

is a covering which refines $\mathcal{U}$ and $\mathcal{V}$. Clearly

$$\bigcap_{g \in G} g(\mathcal{U})$$

is an invariant open covering which refines $\mathcal{U}$. Hence, if X is a compact space then the finite dimensional invariant open coverings are cofinal in the set of all coverings.

We shall call an invariant covering $\mathcal{U}$ a *G-covering* if it satisfies the following:

For $U \in \mathcal{U}$ and $g \in G$, $U \cap gU \neq \emptyset$ implies that $U = gU$.

We shall also call an invariant G-covering $\mathcal{U}$ a *regular G-covering* if its nerve is a regular G-covering; i. e. if it satisfies the following condition for each subgroup H of G:

If $U_0, ..., U_n$ are members of $\mathcal{U}$ and $h_0, ..., h_n$ are in H and if $U_0 \cap ... \cap U_n \neq \emptyset \neq h_0U_0 \cap ... \cap h_nU_n$, then there is an element h of H with $hU_i = h_iU_i$ for all i.

From the above, we have the following proposition:

Proposition 3.2.1 *Let X be a compact G-space, G finite. Then the finite dimensional regular G-coverings of X are cofinal.* $\square$

Let $\mathcal{U}$ and $\mathcal{V}$ be G-coverings such that $\mathcal{V}$ is a refinement of $\mathcal{U}$. Then there exists a refinement projection $p: \mathcal{V} \longrightarrow \mathcal{U}$ which is equivariant. That is,

$$V \subset p(V) \text{ and } p(gV) = gp(V).$$

For, we choose a representative out of each orbit of G on $\mathcal{V}$ and define p arbitrary on these representatives so that $V \subset p(V)$. If $gV = g'V$, where V is one of these representatives, then

$$[g^{-1}g'p(V)] \cap p(V) \supset [g^{-1}g'V] \cap V = V \neq \emptyset$$

so that $gp(V) = g'p(V)$; i.e., the assignment $g(V) \longrightarrow gp(V)$ is well-defined. Thus we can extend the definition by putting

$$p(gV) = gp(V).$$

Therefore, G-coverings, ordered by existence of equivariant refinement maps, form a directed set. This is not true of arbitrary invariant coverings. For this reason we regard G-coverings as the basic notion, rather that invariant coverings.

Let A be a closed subset of the compact set X. If $\mathcal{U}$ is an open covering of X, then $K(\mathcal{U}|A)$ will denote the subcomplex of $K(\mathcal{U})$ consisting of all simplices $(U_0, ..., U_n)$ such that $U_0 \cap ... \cap U_n \cap A \neq \emptyset$. If $\mathcal{V}$ is a refinement of $\mathcal{U}$ and $p : \mathcal{V} \to \mathcal{U}$ is any refinement projection then the induced simplicial map, $\overline{p} : K(\mathcal{V}) \to K(\mathcal{U})$, carries $K(\mathcal{V}|A)$ into $K(\mathcal{U}|A)$. Moreover, for any two refinement projections $p, q : \mathcal{V} \longrightarrow \mathcal{U}$, $\overline{p}$ and $\overline{q}$ are contiguous on $K(\mathcal{V}|A) \to K(\mathcal{U}|A)$ [E-S, pg. 235]. This implies that the induced homology maps and cohomology maps (arbitrary coefficients)

$$H(K(\mathcal{V}), K(\mathcal{V}|A)) \longrightarrow H(K(\mathcal{U}), K(\mathcal{U}|A)),$$

$$H^*(K(\mathcal{U}), K(\mathcal{U}|A)) \longrightarrow H^*(K(\mathcal{V}), K(\mathcal{V}|A))$$

are independent of the choice of the refinement projection. The definition of Čech homology and cohomology may be taken to be

$$\check{H}(X, A) = \varprojlim H(K(\mathcal{U}), K(\mathcal{U}|A)),$$

$$\check{H}^*(X, A) = \varinjlim H^*(K(\mathcal{U}), K(\mathcal{U}|A)),$$

where the limits are taken over all coverings $\mathcal{U}$. Recall that the set of G-coverings is a cofinal system of coverings. Therefore we will consider only G-coverings for the definition of Čech homology and cohomology.

Now we consider the generalization of the Smith sequences. Thus we will also consider only a cyclic group of order p and work with coefficients in Z_p. Let $\mathcal{U}$ be a G-covering of X and consider the Smith special groups

$$H^\rho(K(\mathcal{U}), K(\mathcal{U}|A)),$$

$$H^*_\rho(K(\mathcal{U}), K(\mathcal{U}|A)),$$

where $\rho = \tau^i$, $1 \leq i \leq p-1$. Since any G-covering $\mathcal{V}$ refining $\mathcal{U}$ possesses an equivariant refinement projection, we have the induced maps

$$H^\rho(K(\mathcal{V}), K(\mathcal{V}|A)) \longrightarrow H^\rho(K(\mathcal{U}), K(\mathcal{U}|A)),$$

$$H_*^\rho(K(\mathcal{U}), K(\mathcal{U}|A)) \longrightarrow H_*^\rho(K(\mathcal{V}), K(\mathcal{V}|A)).$$

Any two equivariant refinement projections induce contiguous equivariant simplicial maps [E-S, pg. 235] and the induced chain maps are equivariantly chain homotopic [Bre2, pg. 126]. Hence the above homology maps are independent of the choice of the equivariant refinement projection. Consequently, we can define *Smith special Čech homology (cohomology) groups,*

$$\check{H}^\rho(X, A) = \varprojlim H^\rho(K(\mathcal{U}), K(\mathcal{U}|A)),$$

$$\check{H}_*^\rho(X, A) = \varinjlim H_*^\rho(K(\mathcal{U}), K(\mathcal{U}|A)),$$

where $\mathcal{U}$ ranges over the G-coverings of X.

Since the direct limit functor is exact [Fu, pg. 58], we have the *Smith long exact sequences* for Čech cohomology groups;

$$\ldots \check{H}_\rho^{n+1}(X, A; Z_p) \xleftarrow{\delta^*} \check{H}_{\bar\rho}^n(X, A; Z_p) \oplus \check{H}^n(X^G, A^G; Z_p) \xleftarrow{i^*} \check{H}^n(X, A; Z_p) \xleftarrow{\rho^*} \ldots .$$

In general, the inverse limit functor is not always exact [Fu,pg.63]. But the inverse limit of an exact sequence of finite dimensional vector spaces over a field, Z_p, is exact [E-S, pg. 226]. Therefore if X is compact and A is closed, then we also have the following *Smith long exact sequences* for Čech homology groups, since the finite G-coverings will be cofinal [Proposition 3.2.1]:

$$\ldots \check{H}_{n+1}^\rho(X, A; Z_p) \xrightarrow{\delta^*} \check{H}_n^{\bar\rho}(X, A; Z_p) \oplus \check{H}_n(X^G, A^G; Z_p) \xrightarrow{i^*} \check{H}_n(X, A; Z_p) \xrightarrow{\rho^*} \ldots .$$

<u>*Theorem 3.2.1*</u> *[Bre2, pg. 144]. Let X be a compact G-space, G a cyclic group of prime order p, and let A be a closed invariant subspace. Then*

$$rk\, \check{H}_\rho^n(X, A; Z_p) + \sum_{i \geq n} rk\, \check{H}^i(X^G, A^G; Z_p) \leq \sum_{i \geq n} rk\, \check{H}^i(X, A; Z_p).$$

The corresponding statement is also true for homology. □

3.3 Newman's Theorem

In this section we shall prove a theorem of Newman [N] in a version due to Smith [S2]. This state that a compact Lie group cannot act on a manifold in such a way as to have "uniformly small" orbits.

Lemma 3.3.1 [Bre2, pg. 154]. Let $\pi : X \longrightarrow Y$ be a surjective map, let $B \subset Y$ be closed and put $A = \pi^{-1}B$. Let $\mathcal{U}$ be a covering of X and suppose that the canonical homomorphism

$$H^n(K(\mathcal{U}), K(\mathcal{U}|A)) \to \check{H}^n(X, A)$$

is onto. If there exists a covering $\mathcal{V}$ of Y such that $\pi^{-1}\mathcal{V}$ refines $\mathcal{U}$, then $\pi^* : \check{H}^n(Y, B) \to \check{H}^n(X, A)$ is onto. □

Proposition 3.3.1 [Bre2, pg. 155]. Let X be a compact space and $A \subset X$ a closed subspace. Suppose that $\check{H}^i(X, A; Z) = 0$ for $i > n$ and that $\check{H}^n(X, A; Z) \simeq Z$. Also assume that if C is any proper closed subspace of X, then $\check{H}^n(X, A; Z) \to \check{H}^n(C, C \cap A; Z)$ is trivial. Let $\mathcal{U}$ be any covering of X such that

$$H^n(K(\mathcal{U}), K(\mathcal{U}|A); Z) \to \check{H}^n(X, A; Z)$$

is onto. Then there does not exist an effective action of any compact Lie group on X leaving A invariant and such that each orbit is contained in some member of $\mathcal{U}$. □

Theorem 3.3.1 [Bre2, pg.156]. Let M be a connected topological n-manifold. Then there exists a finite open covering $\mathcal{U}$ of the 1-point compactification of M such that there is no effective action of a compact Lie group on M with each orbit contained in some member of $\mathcal{U}$.

Proof. The case 1: M is orientable. Let X be a 1-point compactification of M, $A = \{\infty\}$. From the Lefschetz duality theorem [Sp, pg. 297; Mu, pg. 415]:

$$H_q(X - A; Z) \simeq \check{H}^{n-q}(X, A; Z),$$

we have $\check{H}^n(X,A;Z) \simeq Z$ and $\check{H}^i(X,A;Z) = 0$ for $i > n$. Let $V \subset M$ be an open n-disk and consider the pair $(X, X-V)$ and $\mathrm{Int}(X-V)$. By the excision theorem for cohomology, the inclusion map, $(X,\ \mathrm{Int}(X-V)) \to (X, X-V)$, induces an isomorphism for cohomology. Also the inclusion $(X,A) \to (X, X-V)$ induces an isomorphism for cohomology. Together, we have

$$\check{H}^n(V,\partial V;Z) \xleftarrow{\cong} \check{H}^n(X,X-V;Z) \xrightarrow{\cong} \check{H}^n(X,A;Z).$$

Now let $C \subset X$ be a closed subset and choose an open n-disk $V \subset X - C$ and consider the following diagram;

$$\begin{tikzcd}
\check{H}^n(X, X-V) \arrow[r, "\cong"] \arrow[d] & \check{H}^n(X, A) \arrow[d] \\
0 = \check{H}^n(C, C) = \check{H}^n(C, C \cap (X-V)) \arrow[r] & \check{H}^n(C, C \cap A))
\end{tikzcd}$$

Then $\check{H}^n(X,A;Z) \to \check{H}^n(C, C\cap A; Z)$ is trivial.

Now we construct an explicit open covering of X which satisfies the condition of Proposition 3.3.1. Let D be an n-disk in M, and let $f : X \to \partial\Delta^{n+1}$ be a map which is a homeomorphism of $\mathrm{Int}(D)$ onto $\partial\Delta^{n+1} - v_0$ and $X-$ $\mathrm{Int}(D)$ onto v_0. See Figure 3.3.1.

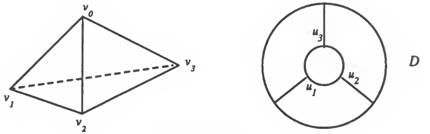

FIGURE 3.3.1

Let $U_i = f^{-1}(\mathrm{st}\ \ v_i)$ for each $i = 0, ..., n+1$ and consider the covering

$$\mathcal{U} = \{U_i | i = 0, ..., n+1\}.$$

Then the natural homomorphism

$$H^n(K(\mathcal{U}), K(\mathcal{U}|A); Z) \to \check{H}^n(X, A; Z),$$

is an isomorphism. Since this covering $\mathcal{U}$ satisfies the condition of Proposition 3.3.1, we proved the case when M is orientable.

The case 2: M is nonorientable. Let L be a polyhedron obtained from $\partial\Delta^{n+1} \cup (w, v_0)$ and let D and D^* be two n-disks in M with $D \subset Int(D^*)$. We define a map $h : M \to L$ by taking Int(D) homeomorphically onto $\partial\Delta^{n+1} - v_0$, taking ∂D to v_0, taking Int$(D^*) - D$ to (w, v_o) and taking $M-$Int(D^*) to w.

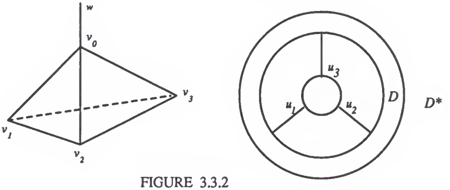

FIGURE 3.3.2

Let $\mathcal{U}$ be the covering of X by inverse images of open stars of L, and let $U_i = h^{-1}(st\, v_i)$ for each i $= 0, ..., n+1$, and $W = h^{-1}(st\, w)$. We claim that no effective action of a compact Lie group on M can have all orbits contained in members of $\mathcal{U}$. Suppose there is such an action of G. $G(D) \cap D \neq \emptyset$ since, for $x \in U_1 \cap U_2 ... \cap U_{n+1}$, $G(x) \in U_i$ for some $i = 1, ..., n+1$. $G(D) \subset Int(D^*.)$ Notice that G is a compact group, hence G is equicontinuous. Therefore there is a connected open neighborhood V of D with $G(V) \subset$ Int(D^*). Since all transformations of V intersect D, $G(V)$ is connected. Therefore we may as well take $V = G(V)$. But V is an orientable n-manifold and

$$\mathcal{U} \cap V = \{U_1, U_2, ...U_{n+1}, W \cap V\}$$

is an open covering of V of the type constructed above for an orientable n-manifold. It follows that G acts trivially on V. This contradicts another Newman's theorem [N], [Dr] which we will also prove. □

Theorem 3.3.2 [N, Dr]. If G is a compact Lie group acting effectively on a connected n-manifold M, then the fixed point set, M^G, is nowhere dense.

Proof. Let D be an n-disk contained in M^G. By lifting the action to the orientable double covering [[Bre1], [Bre2, pg. 67] and [St, pg. 87]] of M in case M is nonorientable, we can assume that M is orientable. Again we define the map $f : M \to \partial\Delta^{n+1}$ as in Theorem 3.3.1. Then every orbit is contained in some U_i, since $U_i \subset M^G$ for $i = 1, ..., n+1$. Now let X be the 1-point compactification of M with $A = \infty$. Then the induced covering of X contradicts the Proposition with $A = \infty$ by the same argument as in the proof of the orientable case of Theorem 3.3.1. □

As a corollary of Theorem 3.3.1 we have *Newman's Theorem.*

Theorem 3.3.3 [N, Dr]. Let M be a connected manifold with metric d. Then there exists an $\epsilon > 0$ such that, for every action of a compact Lie group G on M, there exists an orbit of diameter larger than ϵ.

Proof. Let $U_0, U_1, ..., U_{n+1}$ be the covering of $M \cup \infty$ as in Theorem 3.3.1 with $\infty \in U_0$. Then $M - U_0$ is compact and contained in $U_1 \cup U_2 ... \cup U_{n+1}$. Let $\epsilon > 0$ be such that $B_{2\epsilon}(x) \subset U_i$ for some $i = 1, ..., n+1$. Suppose G acts on M and $diam(G(y)) \leq \epsilon$ for all $y \in M$. Then $G(y) \subset B_\epsilon(y)$ for all $y \in M$. Now we can consider two cases:
(1) If $B_\epsilon(y) - U_0 \neq \emptyset$. We choose $x \in B_\epsilon(y) - U_0$ then $y \in B_\epsilon(x)$ and hence

$$G(y) \subset B_\epsilon(y) \subset B_{2\epsilon}(x) \subset U_i$$

for some $i = 1, ..., n+1$.
(2) If $B_\epsilon(y) - U_0 = \emptyset$ then

$$G(y) \subset B_\epsilon(y) \subset U_0.$$

Consequently each orbit is contained in U_i for some $i = 0, ..., n+1$. This contradicts the choice of the covering $\{U_i\}$, we proved the theorem. □

<u>*Corollary 3.3.1*</u> *Let R^n have the usual Euclidean metric. Then an action of a compact Lie group G on R^n can not have orbits of uniformly bounded diameter.*

Proof. Let $\epsilon > 0$ be as in Theorem 3.3.3 for $M = R^n$, and let N be a bound for the diameters of the orbits of a given action Θ of G on R^n. Define a new action Θ^* by putting

$$\theta_g^*(x) = \epsilon/N\, \theta_g(N/\epsilon\, x).$$

Then

$$\| x - \theta_g^*(x) \| = \epsilon/N \; \| (N/\epsilon\, x) - \theta_g(N/\epsilon\, x) \| \leq \epsilon$$

contrary to Theorem 3.3.3. □

CHAPTER 4
P-ADIC TRANSFORMATION GROUPS ON COMPACT 3-MANIFOLDS

4.1 Almost Periodic Homeomorphisms on Compact Metric Spaces

A homeomorphism h of a metric space (X, d) onto itself is said to be *almost periodic* [A.P.] on X iff, for every $\epsilon > 0$, there exists a relatively dense sequence $\{n_i\}$ of integers (i. e. the gaps are bounded) such that $d(x, h^{n_i}(x)) < \epsilon$ for all $x \in X$ and $i = \pm 1, \pm 2, \pm 3, \ldots$. In particular if, for every $\epsilon > 0$, there exists a positive integer n_ϵ such that $d(x, h^k(x)) < \epsilon$ for all $x \in X$ and for all $k \in n_\epsilon Z$, we say that the homeomorphism h is *regularly almost periodic* [R.A.P.].

Below, we state a well-known characterization [Proposition 4.1.1] and property [Proposition 4.1.2] of almost periodic homeomorphisms.

Proposition 4.1.1 [G2, pg. 341]. Let X be a compact metric space. Then h is an almost periodic homeomorphism on X if and only if the set of powers of h is equicontinuous on X. □

Proposition 4.1.2 [G1, pg. 55]. Let h be an almost periodic homeomorphism on a compact metric space (X, d) and let ϵ be any positive number. Then there exists a regularly almost periodic homeomorphism H on X such that $d(h(x), H(x)) < \epsilon$ for each $x \in X$. The homeomorphism H may be chosen as the uniform limit of a sequence of positive powers of h. □

A homeomorphism h of a metric space (X, d) onto itself is said to be *nearly periodic* [N.P.] iff there exists a complete system $\{\Omega_i\}_{i=1}^{\infty}$ of finite covers which are invariant under h. The sequence $\{\Omega_i\}_{i=1}^{\infty}$ is called a *complete system* iff $\{\text{mesh } (\Omega_i)\}$ has limit 0.

4.2 Nearly Periodic and Regularly Almost Periodic Homeomorphisms

In this section we will prove our first main theorem [Theorem 4.2.1] of this chapter and give an example to show that the compactness hypothesis is necessary. P. A. Smith [S3] shows how to construct a compact 0-dimensional group acting on a compact manifold, M, generated by a given N.P. transformation, T, of M onto itself. He then stated, without proof, that every element of a p-adic transformation group acting on a compact metric space is nearly periodic. In this section, we also provide a proof of this theorem. (See Proposition 4.2.1.)

Theorem 4.2.1 *Regularly almost periodic is equivalent to nearly periodic for homeomorphisms on compact metric spaces.*

Proof. Let h be R.A.P. on a compact metric space X. We shall construct a complete system $\{\Omega_i\}_{i=1}^{\infty}$ of finite covers which are invariant under h and such that $\{\text{mesh } (\Omega_i)\}$ has limit 0.

Let $\epsilon_1 > 0$ and choose a finite open cover

$$B(x_{1,1}, \epsilon_1),\ B(x_{1,2}, \epsilon_1), ..., B(x_{1,k_{\epsilon_1}}, \epsilon_1).$$

Since h is R.A.P. there exists a positive integer n_{ϵ_1} such that $d(x, h^n(x)) < \epsilon_1$ for all $n \in n_{\epsilon_1} Z$. We let powers of h act on each $B_{1,k}$, where $B_{1,k} = B(x_{1,k}, \epsilon_1)$. Set

$$B_{1,k}^l = \bigcup_{j=0}^{\infty} h^{jn_{\epsilon_1}+l}(B_{1,k}), \text{ for } l = 0, 1, 2, ..., n_{\epsilon_1} - 1.$$

Then $\{B_{1,k}^l : l = 0, 1, 2, ..., n_{\epsilon_1} - 1\}$ is invariant under h and

$$\{B_{1,k}^l : l = 0, 1, 2, ..., n_{\epsilon_1} - 1, k = 1, 2, ..., k_{\epsilon_1}\}$$

forms a finite invariant open cover of X, denoted by Ω_1.

Now let $\delta_1' > 0$ be the Lebesgue number corresponding to Ω_1. Since the set of powers of h is equicontinuous on X [Proposition 4.1.1], we can find $\alpha > 0$ such that

$$h^i(B(x, \alpha)) \subset B(h^i(x), \delta_1'), \text{ for } i \in Z \text{ and } x \in X.$$

Let $\delta_1 = min\{\delta_1{}', 1/2^2\}$. Let $\epsilon_2 > 0$ be the number such that

$$h^i(B(x,\epsilon_2)) \subset B(h^i(x), \delta_1/6), \text{ for } i \in Z \text{ and } x \in X$$

and consider the finite open cover

$$\{B(x_{2,k}, \epsilon_2) : k = 1, 2, ..., k_{\epsilon_2}\}.$$

For convenience, we denote $B(x_{2,k}, \epsilon_2) = B_{2,k}$. Also, from the definition of R.A.P., there exists a positive integer n_{ϵ_2} such that $d(x_{2,k}, h^n(x_{2,k}) < \epsilon_2$ for all $n \in n_{\epsilon_2}Z$. We let powers of h act on $B_{2,k}$ for each k and set

$$B^l_{2,k} = \cup_{j=0}^{\infty} h^{jn_{\epsilon_2}+l}(B_{2,k}), \text{ for } l = 0, 1, ..., n_{\epsilon_2} - 1, k = 1, 2, ...k_{\epsilon_2}.$$

Then the family of open sets $B^l_{2,k}$, for $l = 0, 1, ..., n_{\epsilon_2} - 1$, is invariant under h and diam $(B^l_{2,k}) < \delta_1$ for each l. Therefore $B^l_{2,k}$ is contained in $B^{l'}_{1,j}$ for some j, l'. Consequently we get the finite open cover

$$\{B^l_{2,k} : l = 0, 1, ...n_{\epsilon_2} - 1, k = 1, 2, ..., k_{\epsilon_2}\} = \Omega_2$$

which is invariant under h, mesh $(\Omega_2) <$ mesh (Ω_1) and mesh $(\Omega_2) < 1/2^2$.

Inductively, we can get a complete system $\{\Omega_i\}$ of finite open covers which are invariant under h and such that mesh $(\Omega_i) < 1/2^i$.

The converse is clearly true from the definition. □

Theorem 4.2.1 is not true without the compactness assumption as the following example shows. This example also shows that Proposition 4.2.1 is false without the compactness assumption.

Example. Let $\bar{d} = min\{d(a,b), 1\}$ be a metric on the real line and define the metric $\rho(\bar{x}, \bar{y})$ on R^∞ to be the l.u.b.$\{\bar{d}(x_i, y_i)\}$ for $\bar{x} = \{x_1, x_2, ...\}, \bar{y} = \{y_1, y_2, ...\}$. We consider the following tree X which is embedded in R^3 so that the segment [a,b] is located at z-axis with $d(a,b) = 1/4$ and is contained in the unit disc in the plane.

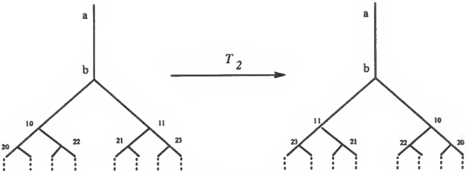

FIGURE 4.2.1

We also define the period 2 homeomorphism T_2 described in Figure 4.2.1, below. (i. e., T_2 is the π rotation about z-axis fixing the segment [a,b]).

We can define a periodic homeomorphism T_{2^i} on X with period 2^i depending on $T_{2^{i-1}}$.

For example, T_{2^2}, which is the the periodic homeomorphism of X with period 4, is the composition of T_2 and the π rotation of subtree below 11, fixing the complement of this subtree and described in Figure 4.2.2, below.

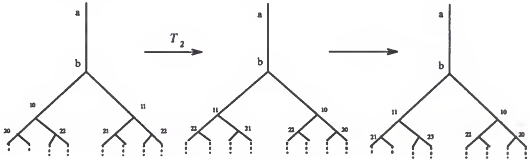

FIGURE 4.2.2

Define

$$T = \lim_{i \to \infty} T_{2^i}.$$

Then T is R.A.P. and N.P. on X. This shows that we can define N.P. on a metric space without compactness. Now we consider the wedge product

$$X^* = \vee_{i=1}^{\infty} X_i \subset \prod (R^3)_i,$$

where X_i is the tree described above, and

$$T^* = \prod_{i=1}^{\infty} T_i : \vee_{i=1}^{\infty} X_i \longrightarrow \vee_{i=1}^{\infty} X_i,$$

where $T_i = T$ on X_i. Notice that we embed X^* in $\prod_{i=1}^{\infty}(R^3)_i$ so that the wedge point, a, is at the origin. Let $x \in X^*$ with $d(x,0) > 1/4$. Then, for sufficiently small $\epsilon > 0$,

$$B_\rho(x,\epsilon) \cap X^* \subset X_j \,\text{for some}\, j.$$

Hence we can not find a complete system of finite open covers of X^*. But T^* is obviously R.A.P. □

Corollary 4.2.1 *The following statements are equivalent on a compact manifold.*

1. *A regularly almost periodic homeomorphism on a compact manifold is periodic.*

2. *A nearly periodic homeomorphism on a compact manifold is periodic.*

3. *(Newman's property on regularly almost periodic homeomorphisms). Let h be a regularly almost periodic homeomorphism of a compact manifold onto itself. Then there exists $\epsilon > 0$ such that every h^i action on X, with $d(x, h^{iZ}(x)) < \epsilon$ for $i \in Z_+, x \in X$, is trivial.* □

Remark. Newman's theorem [N] for periodic homeomorphism on manifolds has also been proved by A. Dress [Dr] and P. A. Smith [S2]. See also [Bre2, p.154]. Later H. T. Ku [K-K-M] extended Newman's theorem to actions of p-adic solenoids.

Proposition 4.2.1 *Let X be a compact metric space and G be a p-adic transformation group acting on X. Then every element of G is nearly periodic, and consequently by Theorem 2.1, also regularly almost periodic on X.*

Proof. We use the technique of Yang [Ya1] . Let X be a compact metric space and G be a p-adic group acting as a topological transformation group on X. Let

$G = G_0 \supset G_1 \supset ...$ be a sequence of open subgroups of G such that, whenever $j > i$, G_i/G_j is a cyclic group of order p^{j-i} and $\lim_{i\to\infty}\{$ diam $(G_i)\}$=0. Let

$$h_{i,j} : G/G_j \longrightarrow G/G_i,\ h_i : G \longrightarrow G/G_i$$

be homomorphisms induced by the identity homomorphism of G; i.e. $gG_j \longrightarrow gG_i$, $g \longrightarrow gG_i$ by $h_{i,j}, h_i$ resp. Then

$$G \simeq \varprojlim\{G/G_j\}$$

with bonding map $h_{i,j}$. Similarly, we let

$$\pi_{i,j} : X/G_j \longrightarrow X/G_i,\ \pi_i : X \longrightarrow X/G_i$$

be maps induced by the identity homeomorphism of X. Then $\{X/G_i : \pi_{i,j}\}$ is an inverse system and $\{\pi_i\}$ determines a homeomorphism of X onto $\lim_{\leftarrow}\{X/G_j\}$ by $x \longrightarrow (xG_1, xG_2...)$. Note that X/G_i denotes the orbit space of X determined by G_i.

Let T be an element of G and for every non-negative integer i, let T_i be the coset TG_i in G/G_i. Since TG_i acts on X/G_i, T_i is a periodic map on X/G_i with period p^k, where $k \leq i$. Let T_j be the periodic map for $j > i$. Then the period of T_j is no less than the period of T_i since $G_i \supset G_j$. We assume that the period of T_1 is p. Let

$$\Psi_1 = \{U_{1,1}, U_{1,2}, ..., U_{1,k_1}\}$$

be a finite open cover of X such that Ψ_1 is π_1-saturated for some finite open cover of X/G_1. We let powers of T_1 act on Ψ_1 and find the finite open cover

$$\Omega_1 = \{\Psi, T_1(\Psi), ..., T_1^{p-1}(\Psi)\}$$

which is invariant under T.

Now let $\delta_1{}'$ be the Lebesgue number of Ω_1, $\delta_1 = min\{\delta_1{}', 1/2^2\}$ and let $\epsilon_2 > 0$ be a number such that

$$T^i(B(x, \epsilon_2)) \subset B(T^i(x), \delta_1/6) \text{ for all } i, \text{ for all } x \in X.$$

Let G_k be a sufficiently small subgroup such that

$$\Psi_k = \{U_{k,1}, U_{k,2}, ..., U_{k,l_k}\}$$

is a finite open cover of X with $\text{diam}(U_{k,i}) < \epsilon_2$ and π_k-saturated for some finite open cover of X/G_k. Then

$$\Omega_k = \{\Psi_k, T_k(\Psi_k), ..., T_k^{q-1}(\Psi_k)\}$$

where q is the period of T_k, is an invariant finite open cover of X under T and mesh $(\Omega_k) < \delta_1$. Thus we get the finite open cover Ω_k which is invariant under T and mesh $(\Omega_k) < 1/2^2$.

Inductively we get a complete system $\{\Omega_i\}$ of finite open covers which are invariant under T and mesh (Ω_i) has limit 0. □

Now we study how a N. P. transformation generates a compact 0-dimensional group on a compact complete metric space. This material is taken from P. A. Smith [S3] .

Let M be a metric space and let $\{\mathcal{U}\}$ be a complete system for the N. P. transformation T acting on M. Let $\mathcal{N}$ be the nerve of $\mathcal{U}$. Then $\mathcal{N}$ is a simplicial complex, we also denote the nerve of $\mathcal{U}$ by $\mathcal{U}$.

Since each complex $\mathcal{U}$ is finite, the simplicial transformation which T induces is periodic in $\mathcal{U}$, say of period $p(\mathcal{U})$. If $\mathcal{U}$, $\mathcal{B}$ are in the system $\{\mathcal{U}\}$ and if $\mathcal{B}$ is a refinement of $\mathcal{U}$, then $p(\mathcal{U})$ must be a divisor of $p(\mathcal{B})$. Now let M be a compact metric space. Then we may assume that

$$\{\mathcal{U}\} = \{\mathcal{U}_i \mid i \in N\}, \text{ where } \mathcal{U}_{i+1} \text{ refines } \mathcal{U}_i,\ i = 1, 2, ...$$

Let $r_i = p(\mathcal{U}_i)$. We call $r_1, r_2, ...$ a period sequence admitted by T.

Now we consider the transformation t_i which T induces in the complex $\mathcal{U}_i$. Then $t_i, t_i^2, ..., t_i^{r_i}$ form a cyclic group G_i of order r_i. The correspondence

$$t_{i+1}^h \to t_i^k \,(h = 0, 1, ..., r_{i+1};\; k = 0, 1, ..., r_i;\; k = h\, mod\; r_i)$$

defines a homomorphism ϕ_i with $\phi_i(t_{i+1}) = t_i$. Let A_{r_i} be the inverse limit group of sequence (G_i, ϕ_i). Then A_{r_i} is a compact topological 0-dimensional group, called r_i-adic group. If $r_i = p^i$ for some p, then A_{r_i} is a so-called p-adic group. Let us now assume that the space M is complete. We will show how, by the adjunction of certain limit transformations to the cyclic transformation group $\mathcal{T}$ generated by T, we obtain a transformation group $\overline{\mathcal{T}}$ which is a realization of A_{r_i}.

An element of A_{r_i} is, by definition, a sequence of the form

$$t = \{t_1^{a_1}, t_2^{a_2}, ...\}\,(0 \le a_i \le p_i - 1;\; a_i = a_{i+1}\, mod\; r_i).$$

We claim that there exists a transformation T_t of M such that the sequence $T^{a_1}, T^{a_2}, ...$ converges uniformly to T_t. Note that, since r_i is a divisor of r_{i+1} we have $a_m = a_n\, mod\; r_n$ if $m > n$. Consequently, T^{a_m} effects precisely the same permutation of the component sets of $\mathcal{U}_n$ as T^{a_n} does if $m > n$. Hence, if we assign to each point x a set U_x^n of $\mathcal{U}_n$ such that $x \in U_x^n$, we have

$$T^{a_m}(x) \in T^{a_m}(U_x^n) = T^{a_n}(U_x^n)\; if\; m > n.$$

Thus if $m, k > n$,

$$d(T^{a_m}(x), T^{a_k}(x)) < mesh(\mathcal{U}_n),$$

and since mesh$(\mathcal{U}_n) \to 0$, the sequence $T^{a_i}(x)\,(i = 1, 2, ...)$ is a Cauchy sequence, and hence converges. Clearly the convergence is uniform, and therefore we obtain a continuous map T_t of M. T_t is homeomorphism and depends continuously on t. In fact, let $x_1, x_2 \in M$. Then there exists $\mathcal{U}_m$ in $\{\mathcal{U}_i | i \in N\}$ such that $x_1 \in U_{m_1}$, $x_2 \in U_{m_2}$ with $U_{m_1} \cap U_{m_2} = \emptyset$. Then $T^{a_m}(U_{m_1}) \cap T^{a_m}(U_{m_2}) = \emptyset$. Therefore $T_t(x_1) \neq$

$T_t(x_2)$, hence T_t is a homeomorphism. Now let $d(T_t, T_s) < \epsilon$ with $t = t_1^{a_1}, t_2^{a_2}, \ldots$ and $s = t_1^{b_1}, t_2^{b_2}, \ldots$. Then there exists n such that $a_i = b_i$ for all $i \leq n$. Therefore we may consider that t, s are in a topologically small subgroup. Consequently T_t depends continuously on t. Moreover, the transformations T_t constitute a group $\overline{T}$ with $T_t T_s = T_{ts}$, where t, s are elements of A_{r_i}, and $\overline{T}$ is a realization of A_{r_i}. Clearly the original N. P. transformation T is T_t where $t = (t_1, t_2, \ldots)$.

Thus, if M is a compact metric space, then every N. P. transformation T is contained in a compact 0-dimensional group $\overline{T}$ of N. P. transformations, $\overline{T}$ is the completion of the group generated by T, and the 0-dimensional group which it realizes is uniquely determined by T.

Conversely, every transformation in a compact 0-dimensional group of transformations of a compact metric space is nearly periodic by the same argument as the proof of Proposition 4.2.1 and Theorem 2.2.1.

4.3 Almost Periodic Homeomorphisms and Hilbert-Smith Conjecture

Newman [N] , P. A. Smith [S2], and A. Dress [Dr], proved that if G is a compact Lie group acting effectively on a topological n-manifold M, then the fixed point set is nowhere dense. From this result we have the following proposition:

Proposition 4.3.1 *Let M be a compact, connected n-manifold with nonempty boundary and let h be a periodic homeomorphism of M onto itself such that h is the identity on ∂M. Then h is the identity on M.*

Proof. We attach two copies of M on their boundaries with the identity map and let H be the map of $M \cup_{id|\partial M} M$ onto itself such that $H = h$ on one copy of M and $H = identity$ on the other copy of M. Then H is periodic on this double of M and the fixed point set contains an open subset. Therefore $h = identity$, by Newman's theorem. □

In general, A.P. and R.A.P. are not equivalent on a metric space . For example, an irrational rotation on S^1 or D^2 is A.P., but is not R.A.P. [F], [vK]. But, from Proposition 4.2.1, Proposition 4.3.1 and Corollary 4.2.1, we have the following theorem:

<u>*Theorem 4.3.1*</u> *Let M be a compact, connected 3-manifold with nonempty boundary and let h be a homeomorphism of M onto itself. Then each of the following is equivalent to the Hilbert-Smith conjecture on M:*

1. *If h is almost periodic on M, with h=identity on ∂M, then h=identity on M.*
2. *If h is regularly almost periodic on M, with h=identity on ∂M, then h=identity on M.*
3. *If h is regularly almost periodic on M, then h is periodic on M.*
4. *If h is nearly periodic on M, then h is periodic on M.*
5. *(Newman's property on regularly almost periodic homeomorphisms). Let h be a regularly almost periodic homeomorphism of M onto itself. Then there exists $\epsilon > 0$ such that every h^i action on M, with $d(x, h^{iZ}(x)) < \epsilon$ for $i \in Z_+$, $x \in M$, is trivial.*

Proof. (1) implies (2): clear.

(2) implies (3): Let h be R.A.P. on M then h is periodic on ∂M [G_1]. Let n be the period of h on ∂M. Then h^n is R.A.P. and identity on ∂M. Therefore h^n is the identity on M, by hypothesis, and hence h is periodic M.

(3) implies (1): Let h be A.P. with $h = identity$ on ∂M, and let $\epsilon > 0$. By Proposition 4.1.2, there exists a R.A.P. homeomorphism H on M such that $d(h(x), H(x)) < \epsilon$, for each $x \in M$. H is periodic by assumption. Since H is a uniform limit of a sequence of positive powers of h and $h = identity$ on ∂M, $H = identity$ on ∂M. Therefore

$H = identity$ on M, by Proposition 4.3.1. Since ϵ was arbitrary, $h = identity$ on M.

By Corollary 4.2.1, (3), (4), and (5) are equivalent.

Clearly (4) implies the Hilbert-Smith conjecture from the Proposition 4.2.1. Conversely let T be N. P. which is not periodic. Then we can construct the compact totally disconnected group G generated by T [S3], and G can act effectively on M. $\square$

APPENDIX A
A THREE DIMENSIONAL PRIME END THEORY

A THREE DIMENSIONAL PRIME END THEORY AND THE HILBERT-SMITH CONJECTURE

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June 14, 1993

Abstract

Prime end theory is essentially a compactification theory for simply connected, bounded domains, U, in E^2, or simply connected domains in S^2 with nondegenerate complement. The planar case was originally due to Caratheodory and was later generalized to the sphere by Ursell and Young, and to arbitrary two manifolds by Mather. There are many applications of the two dimensional theory, including applications to fixed point problems, embedding problems, and homeomorphism (group) actions.

Several constructions of a three dimensional theory appear in the literature, including work by Kaufmann, Mazurkiewicz, and Epstein.

In this paper, we develop a *simple* three dimensional prime end theory for certain open subsets of Euclidean three space. It includes conditions focusing on an "Induced Homeomorphism Theorem", which we believe provides the necessary ingredient for applications. In particular, we obtain some theorems with potential applications to the Hilbert-Smith Conjecture.

A.1 Introduction

Prime end theory is essentially a compactification theory for simply connected, bounded domains, U, in E^2, or simply connected domains in S^2 with nondegenerate complement. The planar case was originally due to Caratheodory [C], and was later generalized to the sphere by Ursell and Young [U-Y], and to arbitrary two manifolds by Mather [Mat]. For each such domain, U, there is given an associated structure of crosscuts, chains of crosscuts, prime ends, and impressions of prime ends. Caratheodory [C] and Ursell and Young [U-Y] proved the following:

Theorem A.1.1 [C]. The prime ends of U are in 1-1 correspondence with the boundary points of the unit disk. That is, the compactification is by a manifold.

Theorem A.1.2 [U-Y]. There is a C-transformation $\phi : U \to Int(D)$ such that ϕ is uniformly continuous on the collection of crosscuts of U, although not necessarily on U.

*Theorem A.1.3 [U-Y]. (***The Induced Homeomorphism Theorem***). Let U be a simply connected domain in the plane, and let $h : Cl(U) \to Cl(U)$ be a homeomorphism. Let $\phi : U \to Int(D)$ be a C-transformation. Then $\phi h \phi^{-1} : Int(D) \to Int(D)$ can be extended to a homeomorphism of D onto itself.*

There are many applications of the two dimensional theory, including applications to fixed point problems, embedding problems, periodic points of homeomorphisms, and homeomorphism (group) action and extension problems. See, for example, [C-L, Mas, Ep2, Br2,3,4, Br-Mau, Br-May, May1,2, Lew, and Mat], among others.

Several constructions of a three dimensional theory appear in the literature, including work by Kaufmann [Kau], Mazurkiewicz [Maz], and Epstein [Ep1]. These papers have not yet had any applications of which we are aware.

In this paper, we develop a *simple* three dimensional prime end theory for certain open subsets of Euclidean three space. It includes conditions not addressed by

any of the above three authors. Our additional conditions focus on an "Induced Homeomorphism Theorem", which we believe provides the necessary ingredient for applications. In particular, we obtain some theorems with potential applications to the Hilbert-Smith Conjecture.

The first author would like to thank John Mayer for interesting discussions on the topic this paper many years ago, and for the invitation to present these results at his AMS Special Session in Knoxville, Tennessee, in March, 1993. She would also like to express her deep gratitude to both Richard Wiegandt for translating significant portions of Kaufman's work and to John Mayer for his translation of Caratheodory's original paper.

A.2 Definition of a Prime End Theory on E^3

The essential ingredients of the planar prime end theory are included in Theorems 1.1, 1.2, and 1.3 quoted above. A satisfactory three dimensional theory should certainly include these. Thus, we *define* a prime end theory for open subsets U of E^3 to be a theory which satisfies these conditions.

A PRIME END THEORY FOR OPEN SUBSETS OF E^3 MUST INCLUDE THE FOLLOWING:

(1) There exists a *prime end structure* on U including crosscuts, chains of crosscuts, prime ends, and impressions of prime ends, for suitable domains, U, such that the prime ends determine a *prime end compactification*, U^* of U, consisting of U plus its prime ends.

(2) There is defined a homeomorphism ϕ from U onto the the interior of some compact three manifold M^3 with nonempty boundary such that *ϕ is uniformly continuous on the collection of crosscuts of U.*

(3) The *prime end compactification of U is homeomorphic to M^3.*

(4) The *Induced Homeomorphism Theorem* holds on M^3. That is, if $h : Cl(U) \to Cl(U)$ is an onto homeomorphism, then $\phi h \phi^{-1} : Int(M^3) \to Int(M^3)$ can be extended to an (induced) homeomorphism of M^3 onto itself.

In §3, we hypothesize that the domains of our theory come from the collection of bounded, connected, $1-ULC$, open subsets U of E^3 such that U is homeomorphic to the interior of some compact 3-manifold M^3 with nonempty boundary. The latter condition is necessary by (3) above. The Whitehead Example, discussed below, shows why the $1-ULC$ condition is necessary.

In §3, we also present our definitions of "U has a prime end structure", of the space U^*, and of "C-transformation". We show that U^* is compact, so that we can indeed call it the "prime end compactification" of U. Our C-transformation is similar to Ursell and Young's [U-Y] C-transformation, and plays the role of the homeomorphism, ϕ, in our definition above. Of course, there are then two major problems: (1) to characterize those domains U which have prime end structures, and (2) to characterize those open subsets of E^3 which admit a C-transformation onto the interior of some compact 3-manifold. These remain open problems at present.

The Whitehead Example. This example illustrates the typical problem that we must avoid in our open sets. It is an example of a connected, simply connected, contractible, proper open subset of S^3 which is not homeomorphic to E^3. It can be constructed as the complement of the intersection in S^3, of "half-twisted", folded tori. Note that the open set is not 1-connected at infinity, since the fundamental group at infinity is infinitely generated. In particular, it is not $1-ULC$ at infinity. In [Hu1], Husch shows that this example does not have a manifold compactification.

Below, we state some well known theorems which provide sufficient conditions for an open 3-manifold to be homeomorphic to the interior of some compact 3-manifold M^3 with nonempty boundary.

Theorem A.2.1 [Ed]. Let U be a contractible open 3-manifold, each of whose compact subsets can be embedded in E^3. If U is 1-connected at infinity, then U is homeomorphic to E^3.

C. T. C. Wall has a related theorem, for which a corollary is:

Theorem A.2.2 [Wa]. If M is an open 3-manifold in E^3 *which is 1-connected at infinity, then M is homeomorphic to E^3.*

Theorem A.2.3 [Hu2]. Let M be a connected, orientable 3-manifold with compact boundary, and one end. The interior of M is homeomorphic to the interior of a compact 3-manifold iff there exists a positive integer n such that every compact subset of M is contained in the interior of a compact 3-manifold M' with connected boundary such that

1. *$\pi_1(M - M')$ is finitely generated,*
2. *$genus(Bd(M')) \leq n$, and*
3. *every contractible 2-sphere in $M - M'$ bounds a 3-cell.*

A.3 A Three Dimensional Prime End Theory for E^3

In this section, we develop a three dimensional prime end theory for a class of domains in E^3. In §3.1, we make the necessary definitions to set up its structure, including the definition of an *admissible domain*; in §3.2, we prove the existence of such a prime end theory for admissible domains; and in §3.3, we define the *bubble domains* and prove that the bubble domains admit C-transformations and therefore are admissible. The proof requires the use of the Topological Dehn's Lemma, due to Repovs [Re]. We also give some examples of bubble domains to show that they form a large class of interesting domains in E^3.

STANDING HYPOTHESIS FOR §3: The open set U is a bounded, connected, $1 - ULC$, domain in E^3 which is homeomorphic to the interior of some compact 3-manifold M^3.

A.3.1 Definitions

1. A **crosscap** is an open two-cell D in U such that

(1) D separates U into exactly two complementary domains,

(2) $Cl(D)$ is a two-cell, and

(3) $Cl(D) \cap Bd(U) = Bd(D)$.

2. A **chain of crosscaps** in U is a sequence $\{D_i\}_{i=1}^{\infty}$ of crosscaps such that

(1) D_{i+1} separates D_i from $\{D_{i+j}\}_{j=2}^{\infty}$;

(2) $Cl(D_i) \cap Cl(D_j) = \phi$; and

(3) $lim_{i \to \infty}(diam(D_i)) = 0$.

3. Two chains of crosscaps, $\{Q_i\}$ and $\{R_i\}$, are **equivalent** iff

(1) For each Q_i, there exists $j > i$ such that Q_{i+1} separates Q_i from $Q_j \cup R_j$;

(2) For each R_i, there exists $j > i$ such that R_{i+1} separates R_i from $R_j \cup Q_j$. That is, two subsequences can be alternated or "interspersed" to form a new, equivalent chain of crosscaps.

4. A **prime end** of U is an equivalence class of chains of crosscaps of U.

5. Let $\{Q_i\}$ be a chain of crosscaps representing the prime end E of the domain U, and let U_i be the **associated** or **corresponding complementary domain** of Q_i; that is, that complementary domain of Q_i in U which contains $\bigcup\{Q_j\}_{j>i}$. We call the set $Cl(U_i)$ the **corresponding continuum**, and the set $(Cl(U_i)) \cap Bd(U)$ the **corresponding boundary compactum**. The **impression of** E, denoted by $I(E)$, is defined to be the set $\bigcap_i\{Cl(U_i)\}$. Clearly, $I(E) \subset Bd(U)$. If $\{Q_i\}$ converges to a single point x in $Bd(U)$, then x is called a **principal point** of E. As in the two dimensional theory, the set of principal points of E may be nondegenerate. However, if E is a prime end of a compact manifold, then E has exactly one principal point.

6. An *onto* homeomorphism $\phi : U \to Int(M^3)$, where M^3 is a compact 3-manifold, is called a C-**transformation** iff

(1) The image of every chain of crosscaps of U is a chain of crosscaps of $Int(M^3)$. In particular, the image of each crosscap of U is a crosscap of $Int(M^3)$.

(2) On each crosscap Q, ϕ extends to a homeomorphism from $Cl(Q)$ onto $Cl(\phi(Q))$. (However, ϕ does not necessarily extend to a homeomorphism from the union of the closures of all the crosscaps of U to the union of the closures of their images in $\phi(Q)$.)

(3) For each crosscap Q_i of a prime end of U, let U_i be its corresponding domain. Let (Q_i', U_i') be the image of (Q_i, U_i) under ϕ. We consider the following open sets on $Bd(M^3)$: $Int[Cl(U_i') \cap Bd(M^3)]$. We require that the collection of all such open disks on $Bd(M^3)$ form a basis for the topology of $Bd(M^3)$.

7. We say that a domain U **has a prime end structure** iff for every $\epsilon > 0$ there exist a finite number of prime ends, $\{E_i\}_{i=1}^n$, of U, and a finite number of crosscaps, $\{Q_i\}_{i=1}^n$, with Q_i a crosscap of some chain representing E_i, such that

(1) $diam(Q_i) < \epsilon$, and

(2) If U_i denotes the corresponding domain for Q_i, then $Bd(U) \cup \bigcup_{i=1}^n (U_i)$ is a neighborhood of $Bd(U)$ in $Cl(U)$.

8. U is an **admissible domain** iff there exists a C-transformation $\phi : U \to Int(M^3)$, for some compact 3-manifold, M^3. (Recall that U also satisfies the Standing Hypothesis.) The triple (U, ϕ, M^3) is called an **admissible triple.**

A.3.2 Existence of a Prime End Theory on Admissible Domains

This section is divided into four parts, establishing the properties corresponding, respectively, to the four parts of the definition of a prime end theory. In constructing the prime end compactification, U^*, we assume only that U is bounded and that it has a prime end structure. For the remainder of this section, we also assume that U is an admissible domain, which includes, in particular, the existence of a C-transformation, ϕ, taking U onto the interior of some compact 3-manifold, M^3.

The Prime End Compactification, U^*

The *raison d'être* of prime end theory is to use the prime ends as a compactification of the domain in question. Thus, we first define the space U^*. Then, in Theorem 3.1 below, we show that if U is a bounded domain which has a prime end structure, then U^* is compact. Of course, the question arises as to the *existence* of domains in E^3 with a prime end structure. In Theorem 3.2, we show that admissible domains have such structures.

To this end, let U be a bounded domain in E^3 which has a prime end structure, and let U plus the prime ends of U be denoted by U^*. We topologize U^* by declaring the topology of U^* to be generated by basic neighborhoods of the form described below:

The basic neighborhoods of a point of U are the same as the basic neighborhoods of that point in the topology of E^3. Now let E be a prime end of U, and let $\{Q_i\}$ be any chain of crosscaps representing E. Then a basic neighborhood of the prime end E is the corresponding domain W_i of any one of these crosscaps Q_i, plus all the prime ends of U represented by chains of crosscaps which are eventually in W_i.

Theorem A.3.1 Let U be a $1-ULC$ bounded domain in E^3 which has a prime end structure. Then U^* is compact.

Proof. Let $\mathcal{U}^*$ be a basic open cover of U^*. Since U has a prime end structure, for every $\epsilon > 0$, there is a finite set of crosscaps, $\{Q_i\}_{i=1}^n$, coming from a finite set of prime ends, $\{E_i\}$, respectively, such that $diam(Q_i) < \epsilon$ and $\bigcup_{i=1}^n \{Q_i\} \cup Bd(U)$ forms a neighborhood of $Bd(U)$ in $Cl(U)$. (Note that $diam(W_i)$ is not necessarily less than ϵ, where W_i is the corresponding domain of Q_i.)

We claim that there is $\epsilon_* > 0$ such that each such Q_i for that ϵ_*, lies in some $V \in \mathcal{U}$. For suppose not. Let $\epsilon_i \to 0$, and let $Q_{i_1}, \ldots, Q_{i_{n_i}}$ be a finite set of crosscaps of U with corresponding domains $W_{i_1}, \ldots, W_{i_{n_i}}$, respectively, such that each of these

crosscaps has diameter $< \epsilon_i$. For ϵ_1, there is at least one of these crosscaps, say Q_{1_*}, which is not a subset of any element of $\mathcal{U}$. Now $S = Bd(Q_{1_*})$ is a simple closed curve on $Bd(U)$ which is accessible from U. Thus, for each point of S, there is an element of $\mathcal{U}$ whose corresponding domain is a "neighborhood" of this point. Since S is compact, a finite number of such basic open sets in $\mathcal{U}$ cover S. Their crosscaps, whose boundary simple closed curves lie in $Bd(U)$, can be used to define a (small) simple closed curve which collars into U with its collar lying in W_{1_*}, the corresponding domain of Q_{1_*}. Since U is $1-ULC$, this can be completed to a small crosscap Q in U such that $Cl(Q) \cap Cl(Q_{1_*}) = \emptyset$, with $Q \subset W_{1_*}$. Now W_{1_*} union the prime ends that lie in W_{1_*}, cannot be covered by a finite subcollection of $\mathcal{U}$. Thus we can repeat the above process, finding a crosscap Q_{2_*} of diameter $< \epsilon_2$, such that $Q_{2_*} \subset W$, where W is the corresponding domain of Q. Then $Cl(Q_{1_*}) \cap Cl(Q_{2_*}) = \emptyset$ and $Q_{2_*} \subset U_{1_*}$.

Continuing inductively, we can find a chain of crosscaps $\{Q_{i_*}\}$ such that none of these lies in any element of $\mathcal{U}$. But this chain of crosscaps of U defines a prime end, say F, of U, such that F is not an element of any member of $\mathcal{U}$. This contradicts the fact that $\mathcal{U}$ is a cover of U^*, and completes the proof of our claim.

It follows that for some $\epsilon_* > 0$, there is a finite collection $\{V_j\}_{j=1}^n$ of the V's, with $diam(V_j) < \epsilon_*$ for each j, and whose union contains all of the prime ends of U. Let $V^* = \bigcup_{j=1}^n \{V_j\} \cup Bd(U)$. Then V^* forms a neighborhood of $Bd(U)$. Therefore $U - V^*$ is a compact subset, X, of U, so some (other) finite subcollection of $\mathcal{U}^*$ covers X. Then the union of these two finite subcollections of $\mathcal{U}^*$ is a finite subcollection covering U^*. □

Theorem A.3.2 Let U be ANY *domain in E^3 which admits a C-transformation, ϕ. Then U has a prime end structure induced by ϕ.*

Proof. Let $\phi : U \to M^3$ be a C-transformation from U onto the interior of a compact 3-manifold, and let $\epsilon > 0$. By Property (3) of the definition of C-transformation, at each point y of $Bd(M^3)$, there is a crosscap Q'_y of $Int(M^3)$ such that (1) $diam(Q'_y) < \epsilon$, (2) $Q'_y = \phi(Q_y)$, where $diam(Q_y) < \epsilon$ and (3) Q_y belongs to some chain of crosscuts representing a prime end of U. Since M^3 is a manifold, and its boundary is compact, there is a finite subset of these crosscaps $\{Q'_i\}_{i=1}^n$ such that $Bd(M^3) \cup \bigcup\{U'_i\}_{i=1}^n$ forms a neighborhood of $Bd(M^3)$ in M^3. Then the collection $\{Q_i\}_{i=1}^n$ forms the required finite collection in U. □

<u>*Corollary A.3.1*</u> *Let (U, ϕ, M^3) be an admissible triple. Then U has a prime end structure induced by ϕ.*

Proof. An admissible domain admits a C-transformation, by definition. □

<u>C-transformations and Uniform Continuity</u>

In Theorem 3.3 below, we show that a C-transformation is uniformly continuous on the *collection of crosscaps* of U.

The reader should note that, in general, it is *not necessarily* uniformly continuous on all of U. A simple example shows why: Let the domain U be the open unit cube in E^3 minus a two dimensional disk with a portion of its boundary in the boundary of the cube. Then U^* is a 3-cell, but it splits apart the interior of the aforementioned disk into two disjoint open disks. Thus, if p is a point of the interior of that disk, it becomes two points in the prime end compactification of U, so that a small neighborhood of p in the original space, when intersected with U, becomes large in diameter when viewed in U^*. See Figure 3.1 below.

However, we also obtain a corollary to Theorem 3.3 showing that a C-transformation *is* uniformly continuous on all of its domain, when that domain is the interior of a compact 3-manifold.

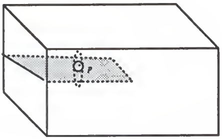


FIGURE A.3.1

Lemma A.3.1 *Let X be a nondegenerate continuum on a compact 2-manifold M. Suppose that $\{X \cap Int(D_i)\}$ is a basis of open sets for X, where each D_i is a disk in M. Then there exists D_{j*} such that $Bd(D_{j*})$ contains at least 2 points of X.*

Proof. Suppose that, for every D_i, $X \cap Bd(D_i)$ is empty or one point. For sufficiently small D_i, $X \cap Bd(D_i) \neq \emptyset$ since X is connected. So now we suppose $X \cap Bd(D_i)$ is one point. Let $p, q \in X$ and let D_j be a neighborhood of p, with $diam(D_j) < \frac{1}{2} d(p,q)$. If $X \cap Bd(D_j) = r$, then r separates p from q. Therefore each pair of points of X is separated by a third point of X, so that the continuum X is a dendrite [Wh, pg 88-89]. Thus X contains an arc. This contradicts the assumption that for every D_i, $X \cap Bd(D_i)$ is one point, since a sufficiently small basic neighborhood of a point of the arc will intersect at least two points. □

Theorem A.3.3 *Let $\phi : U \to Int(M^3)$ be a C-transformation. Then ϕ is uniformly continuous on the collection of crosscaps of U.*

Proof. Suppose that ϕ is not uniformly continuous on the collection of crosscaps of U. Then there exists $\epsilon > 0$ such that for every $\delta > 0$, there exists a crosscap Q_δ of diameter less than δ with $diam(\phi(Q_\delta)) \geq \epsilon$. In particular, for this $\epsilon > 0$, there exist a sequence of positive numbers $\{\delta_i\}$ with $\delta_i \to 0$ and a sequence of crosscaps $\{Q_i\}$ with $diam(Q_i) < \delta_i$ and such that $diam(\phi(Q_i)) \geq \epsilon$. By Whyburn [Wh, pg.11], there is a convergent subsequence $\{Q_i'\}$ of $\{Q_i\}$ such that $lim\,(Q_i') = m \in Bd(U)$; and

there is also a subsequence $\{Q_i''\}$ of $\{Q_i'\}$ such that $\{\phi(Q_i'')\}$ converges to a limit continuum in M^3 (see [Wh, pg. 14]).

Hence, without loss of generality, we may assume that $\{Q_i\}$ converges to a point m of $Bd(U)$ and that $\phi(\{Q_i\})$ converges to a limit continuum X in $Bd(M^3)$, with $diam(X) \geq \epsilon$.

For each $x \in Bd(M^3)$ and each α such that $x \in Int[(Cl(\phi(U_\alpha))) \cap Bd(M^3)]$, let $B_\alpha(x)$ denote $Int[(Cl(\phi(U_\alpha))) \cap Bd(M^3)]$. Now we take $x \in X$ and $B_\alpha(x)$ on $Bd(M^3)$ such that $X \cap Bd(B_\alpha(x))$ contains at least two points p, q. We can do this by Lemma 3.1 and Condition (3) of the definition of C-transformation. Let $\gamma = dist(p, q)$. We consider two sequences $\{p_1, p_2, ...\}$, $\{q_1, q_2, ...\}$, where $p_i, q_i \in \phi(Q_i)$ such that $p_i \to p$, $q_i \to q$. Then there exists an integer N such that for $n > N$, $dist(p_n, q_n) > \frac{\gamma}{2}$. Now the preimages of the sequences $\{p_i\}, \{q_i\}$ converge to the same point, $m \in Bd(U)$. Since M^3 is a manifold, it admits a prime end structure. Thus, let $R_1, \ldots, R_k$ be a finite collection of crosscaps to the boundary of $Int(M^3)$ with corresponding continua each of diameter $< \frac{\gamma}{2}$, and such that the interiors of the corresponding boundary compacta form an open cover of $Bd(M^3)$. Then the collection $\{\phi^{-1}(R_i)\}_{i=1}^{k}$ forms (together with $Bd(U)$) a neighborhood of $Bd(U)$. It follows that infinitely many members of the collection $\{Q_i\}$ lie in one corresponding domain, say the corresponding domain of $\phi^{-1}(R_s)$. Now the image of this corresponding domain has diameter $< \frac{\gamma}{2}$, but infinitely many pairs of points of this image are separated by a distance of at least $\frac{\gamma}{2}$. This contradiction completes the proof. □

<u>Corollary A.3.2</u> *The map ϕ of the interior of the compact 3-manifold M^3 onto itself is a C-transformation iff each of the following three* equivalent *conditions hold:*

(1) ϕ is uniformly continuous on the collection of all crosscuts of $Int(M^3)$,

(2) ϕ can be extended to a homeomorphism $\overline{\phi}$ of M^3 onto itself, and

(3) $\overline{\phi}$ is uniformly continuous on all of M^3.

Proof. (1) This is a direct corollary of Theorem 3.3.

(2) Let $x \in Bd(M^3)$, and let $\{Q_i\}$ be a chain of crosscaps with $lim\ Q_i = x$. Define $\phi(x) = lim\ \phi(Q_i)$. Since a C-transformation takes chains of crosscaps to chains of crosscaps, ϕ is well defined. Now let $\{x_i\}$ be a sequence in $Bd(M^3)$ such that $x_i \to x$, and let $\{Q_{i_j}\}_j$ be a chain of crosscaps of mesh less than $1/2^i$ which converges to x_i. Since $\{Q_{i_j}\}_j$ determines a basis at x_i on $Bd(M^3)$, and $\{Q_i\}$ determines a basis at x on $Bd(M^3)$, and $x_i \to x$, it follows that $\phi(x_i) \to \phi(x)$. Therefore ϕ has a (unique) extension to a homeomorphism of M^3 onto itself.

(3) This is a corollary of (2) above.

Clearly, (3) implies (1), so that conditions (1) to (3) are all equivalent. Further, (2) implies that ϕ is a C-transformation. Thus, all are equivalent to the statement that ϕ is a C-transformation. □

Manifold Compactifications

It is clear that the boundary points of a compact manifold with nonempty boundary are naturally in one-to-one correspondence with the prime ends of the interior of that manifold. In the following, we first show that there is a one-to-one correspondence between the prime ends of an admissible domain, U, and the prime ends of the interior of a compact 3-manifold, M^3, with nonempty boundary. Our main theorem (3.5) of this part then follows easily. Thus, for admissible domains, the prime end compactification of U is indeed a manifold compactification, and we can think of the C-transformation as a "compactification map".

Theorem A.3.4 *Let ϕ be a C-transformation, $\phi : U \to Int(M^3)$. Then ϕ determines a one to one correspondence between the prime ends of the domain U and the prime ends of $Int(M^3)$.*

Proof. Let E be a prime end of U. Then $\phi(E)$ is a prime end of $Int\,(M^3)$, by the definition of C-transformation. We suppose that E_x and E_y are prime ends of U with

$\phi(E_x) = \phi(E_y)$ and we take chains of cross caps $\{E'_{x,i}\}$ and $\{E'_{y,i}\}$ as representatives of $\phi(E_x)$, $\phi(E_y)$, respectively. Let $E'_{x,i} = \phi(E_{x,i})$ and $E'_{y,i} = \phi(E_{y,i})$, so that $\{E_{x,i}\}$ and $\{E_{y,i}\}$ are representatives of E_x and E_y respectively. Now we construct a new chain of crosscaps of $Int\,(M^3)$, by forming an alternating sequence from the chains $\{E'_{x,i}\}$ and $\{E'_{y,i}\}$:

$$\{E'_{x,i_1}, E'_{y,i_1}, E'_{x,i_2}, E'_{y,i_2}, \ldots\}$$

which is equivalent to each of $\{E'_{x,i}\}$ and $\{E'_{y,i}\}$. Then

$$\{E_{x,i_1}, E_{y,i_1}, E_{x,i_2}, E_{y,i_2}, \ldots\}$$

forms a chain of cross caps of U and is clearly equivalent to each of $\{E_{x,i}\}$ and $\{E_{y,i}\}$. Thus, $E_x = E_y$ and ϕ is one-to-one.

To show that this correspondence is onto, let F be a prime end of $Int(M^3)$ and let p be the principal point of F. We take $B_1(p)$. By Theorem 3.3, there exists δ_1 such that if Q is a crosscap of U with $diam(Q) < \delta_1$, then $diam(\phi(Q)) < 1/2$. We notice that the set of images of all crosscaps with diameter less than δ_1, determines a basis for $Bd(M^3)$ in the sense of Condition (3) of the definition of C-transformation. Therefore we can find Q_1 such that:

(i) the "small" complementary domain of $\phi(Q_1)$ is contained in $B_1(p)$, and

(ii) $Int[Cl(\phi(U_1)) \cap Bd(M^3)]$ is a neighborhood of p.

Now we take ϵ'_2 so that $B_{\epsilon'_2}(p) \subset Int(Cl(\phi(U_1)))$. Let $\epsilon_2 = min(1/4, \epsilon'_2)$ and consider $B_{\epsilon_2}(p)$. Again by Theorem 3.3, given $\epsilon_2/2$, there exists δ'_2 such that if Q is a crosscap of U with $diam(Q) < \delta'_2$, then $diam(\phi(Q)) < \epsilon_2/2$. Let $\delta_2 = min\,\{\delta'_2, \delta_1, 1/4\}$. As above, we notice that the set of images of all crosscaps with diameter less than δ_2, determines a basis for $Bd(M^3)$ in the sense of Condition (3) of the definition of C-transformation. Therefore we can find Q_2 such that:

(i) the "small" complementary domain of $\phi(Q_2)$ is contained in $B_{\epsilon_2}(p)$, and

(ii) $Int[Cl(\phi(U_2)) \cap Bd(M^3)]$ is a neighborhood of p.

Continuing inductively, we get a chain of crosscaps $\{Q_i\}$ such that $\{\phi(Q_i)\}$ is equivalent to F. It follows that ϕ is onto. □

<u>Theorem A.3.5</u> *Let U be an admissible domain. Then U^* is homeomorphic to a compact 3-manifold-with-nonempty-boundary, M^3, in such a way that U is identified with $Int(M^3)$ and the prime ends of U are identified with $Bd(M^3)$. Thus, the prime end compactification of U is a manifold compactification.*

Proof. Since U is admissible, there exists a C-transformation, ϕ, taking U onto the interior of some compact 3-manifold, M^3. ϕ is one-to-one on U by definition, and by Theorem 3.4, ϕ determines a one-to-one correspondence between the prime ends of the respective domains. We need only show that ϕ is continuous at the prime ends of U.

Let W be a neighborhood of some prime end of M^3 (that is, of some point p in $Bd(M^3)$). By Property (3) of the definition of C-transformation, there exists a crosscap Q_α of U, with corresponding domain U_α, such that $\phi(Q_\alpha \cup U_\alpha) \subset W$. By the definition of the topology of U^*, U_α plus the prime ends of U_α forms an open set of U^*, and since ϕ takes this set into W, ϕ is continuous on U^*.

Thus, the C-transformation ϕ induces a homeomorphism between U^* and all of M^3. That is, our prime end compactification of the admissible domain U is a manifold compactification. □

<u>The Induced Homeomorphism Theorem</u>

In our view, this is the most important theorem of our three dimensional prime end theory, since it is this theorem which gives rise to our present and anticipated applications.

<u>Theorem A.3.6</u> *Let (U, ϕ, M^3) be an admissible triple (that is, U is an admissible domain and ϕ is its associated map to M^3). If h is a homeomorphism of $Cl(U)$*

onto itself then the induced homeomorphism, $\phi h \phi^{-1}$, of $Int(M^3)$ onto itself can be extended to a homeomorphism $\overline{\phi h \phi^{-1}}$ of all of M^3 onto itself.

Proof. First we show that for a prime end E of $Int(M^3)$, $\phi h \phi^{-1}(E)$ is also a prime end of $Int(M^3)$. Let E be a prime end in $Int(M^3)$ and let $\{E_i\}$ be a representative chain of crosscaps of E. Then there exists a chain of crosscaps $\{D_i\}$ in U such that $\{\phi(D_i)\}$ is equivalent to $\{E_i\}$, by the definition of C-transformation and Theorem 3.3. Then, even though $\{\phi^{-1}(E_i)\}$ may not be a *chain* of crosscaps in U (that is, their diameters may not tend to 0), $\{D_i\}$ is equivalent to $\{\phi^{-1}(E_i)\}$ in the sense that there are subsequences of the crosscaps which alternate. Since h is uniformly continuous, $\{h(D_i)\}$ is also a chain of cross caps of U; and $\{\phi h(D_i)\}$ is a chain of cross caps in $Int(M^3)$ since ϕ is a C-map. But then $\{\phi h(D_i)\}$ is equivalent to $\{\phi h(\phi^{-1}(E_i))\}$ which is the same as $\{\phi h \phi^{-1}(E_i)\}$. Since $\{h(D_i)\}$ is a chain of crosscaps of U, $\{\phi(h(D_i))\}$ is a chain of crosscaps of $Int(M^3)$. Now small crosscaps of a *manifold* cut off small corresponding domains on that manifold. Thus, since $\{\phi h(D_i)\}$ is equivalent to $\{\phi h \phi^{-1}(E_i)\}$, the latter sequence has diameters tending to 0, so that it forms a *chain* of crosscaps of $Int(M^3)$. Thus, $\phi h \phi^{-1}(E)$ is a prime end of $Int(M^3)$.

We notice that the inverse of $\phi h \phi^{-1}$, namely $\phi h^{-1} \phi^{-1}$, behaves in the same way. Therefore $\phi h \phi^{-1}$ acts in a one to one and onto manner, taking prime ends to prime ends. But the prime ends of M correspond precisely to the points of $Bd(M)$.

To show that $\phi h \phi^{-1}$ is extendable to a continuous function, let $x \in Bd(M^3)$ and $\{E_i\}$ be a chain of crosscaps with $lim\, E_i = x$. Define $\overline{\phi h \phi^{-1}}(x) = lim(\phi h \phi^{-1}(E_i))$. Then $\overline{\phi h \phi^{-1}}$ is well defined, since for a prime end E of $Int(M^3)$, $\phi h \phi^{-1}(E)$ is also a prime end of $Int(M^3)$, and each chain of crosscaps representing a prime end of M^3 converges to a point of $Bd(M^3)$. Now if $\{x_i\}$ is a sequence in $Bd(M^3)$ such that $x_i \to x$, then Property (3) of the definition of a C-transformation guarantees that $\overline{\phi h \phi^{-1}}(x_i) \to \overline{\phi h \phi^{-1}}(x)$. It follows that $\phi h \phi^{-1}$ can be extended to the homeomorphism $\overline{\phi h \phi^{-1}}$ of M^3 onto itself. □

A.3.3 Bubble Domains.

In this part, we establish the existence of a nontrivial, interesting class of admissible domains. We define, and give examples of, the "bubble domains", and we prove that these domains are admissible.

A **bubble domain** in E^3 is a bounded, connected, $1-ULC$ open subset U, which is homeomorphic to the interior of some compact 3-manifold, M^3, and whose boundary contains a dense subset S, such that the following conditions are satisfied:

1. There is a monotone map $f : Cl(U) \to M^3$, such that

(1) $f|U$ is a homeomorphism onto $Int\,(M^3)$, where M^3 is a compact 3-manifold with nonempty boundary in E^3, and

(2) for each $x \in S$, $f^{-1}f(x) = x$.

2. $Bd(M^3)$ admits a decreasing sequence of triangulations $\{T_i\}$, with $mesh\,(T_i) \to 0$, such that

(1) the one-skeleton of T_i lies in f(S), and

(2) the boundary of each two-simplex of T_i in $Bd(M^3)$ has inverse of diameter less than $\frac{1}{2^i}$. We call f^{-1}(1-skeleton of T_i) a **1-dimensional ϵ-triangulation of** $Bd(U)$, if $\frac{1}{2^i} < \epsilon$.

3. S is collared into U; that is, there is a homeomorphism $g : S \times [0,1) \to U$ such that

(1) $g(s,0) = s$, for all $s \in S$, and

(2) $g(s,t) \in U$, whenever $t > 0$.

In this situation, U is called a **bubble domain** and $S^3 - U$ is called a **bubble continuum**. The triple (U, f, M^3) is called a **bubble triple**. (We use the word "bubble" because the crosscaps to the boundary look like bubbles on the boundary.)

Two interesting examples are the "bowling ball" and "bowling glove" examples (Figure 3.2), which were constructed by John Mayer. Note that the bowling ball is not an admissible space since it is not 1-ULC.

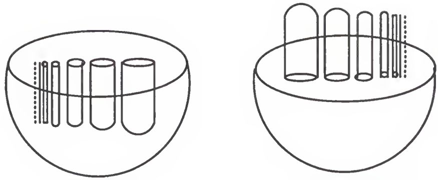


FIGURE A.3.2

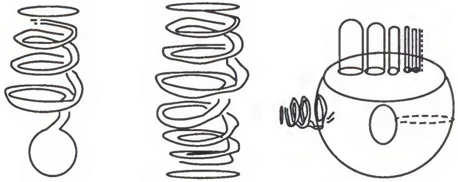
These are basically three dimensional versions of the Warsaw circle, again with a limit segment. The bowling ball (with infinitely many fingers) is a bubble continuum, since its complement is a bubble domain. However, the interior of the bowling ball is not a bubble domain (so its complement, the bowling glove, is not a bubble continuum).

We note that the interior of the bowling ball is not $1 - ULC$, and that there cannot be a C-transformation from the interior of the bowling ball onto the interior of B^3. This is because any sequence of cross caps in the interior of the bowling ball and around the limit segment is necessarily of large diameter. But such a sequence which converges to the limit segment would have to map to a *chain of crosscaps* of $Int(M^3)$ under any C-map.

We construct below a number of other exotic examples (Figure 3.3). Thus, we see that even the very restrictive definition of *bubble domain* gives rise to many examples to which our theory applies.

Before proving that a bubble triple is an admissible triple, we will briefly review Dehn's Lemma which will play an important role in our three dimensional prime end theory.

In 1910, Max Dehn [De] first presented the lemma with a "proof"; however, in 1929 in [Kn], H. Kneser discovered a serious gap in the proof gived by Dehn.



Spiral Double Spiral Distorted Torus

FIGURE A.3.3

The following is the statement of Dehn's Lemma, as given by Bing [Bi2, pg 198]:

Dehn's Lemma: *If D is a PL singular disk in a PL 3-manifold-without-boundary M^3 such that $S(D) \cap Bd(D) = \emptyset$, then there exists a nonsingular PL disk, D_0, such that $D_0 \subset M^3$ and $Bd(D_0) = Bd(D)$.*

In 1957, C. D. Papakyriakopoulos [Pa] proved the lemma. Since Papakyriakopoulos' proof of the lemma, there have been several generalizations and simplifications, including papers by A. Shapiro and J. H. C. Whitehead [S-H] and D.W. Henderson [He].

In 1988, there was a remarkable improvement of Dehn's Lemma by D. Repovs [Re]. In his paper, he extended the classical Dehn's Lemma to a topological version. He only required the map to be continuous and therefore Dehn's disk is a continuous Dehn's disk. That is, a continuous map $f : D^2 \to M^3$ of a two dimensional disk, D^2, into a 3-manifold-without-boundary, M^3, is a Dehn disk iff $S_f \cap Bd(D^2) = \emptyset$, where $S_f = Cl\{x \in D^2 | f^{-1}f(x) \neq x\}$ is the *singular set* of f.

Topological Dehn's Lemma: *Suppose $f : D^2 \to M^3$ is a Dehn disk in a 3-manifold with boundary M^3. Then, for every neighborhood $U \subset M^3$ of $f(S_f)$, there is an embedding $F : D^2 \to M^3$ such that $F|Bd(D^2) = f|Bd(D^2)$ and $F(D^2) - U = f(D^2) - U$.*

Lemma A.3.2 *Let M be a compact 2-manifold without boundary. Then there exists $\epsilon > 0$ such that if T is a triangulation of M with mesh of each simplex less than ϵ, then $Cl(st(v,T))$ is a 2-cell for all $v \in T$.*$\square$

Lemma A.3.3 *Let M^3 be an orientable 3-manifold, and let $Bd(M^3)$, T, and ϵ satisfy the hypotheses of Lemma 3.2 above. Since there are only a finite number of vertices v in T, the collection $\{Bd(Cl(st(v,T)))\}_{v \in K^0}$ forms a finite number of simple closed curves, say $\{C_i\}$, whose interiors form a finite open cover of $Bd(M^3)$. Let Q_i be a crosscap to $Bd(M^3)$ in $Int(U)$, such that $Bd(Q_i) = C_i$. Then the union of the corresponding domains of $\{Q_i\}$ plus $Bd(M^3)$ forms a neighborhood of $Bd(M^3)$ in M^3.* $\square$

In the next two theorems, we prove that a bubble domain is admissible, by showing that it admits a C-transformation.

Theorem A.3.7 *If (U, f, M^3) is a bubble triple, then f induces a prime end structure on U.*

Proof. The proof uses ideas from Bing [Bi1], and the Topological Dehn's Lemma [Re].

We notice that $Int(M^3)$ has the prime end structure induced by the triangulations $\{T_i\}$ of $Bd(M^3)$ from the definition of bubble pair. Let K_i^1 be the 1-skeleton of the triangulation T_i so that $K_i^1 \subset f(S)$, by the definition of bubble domain. Then $S \supseteq \bigcup_{i=1}^{\infty}\{f^{-1}(K_i^1)\}$.

Let $\{\epsilon_i\}$ be a sequence of positive numbers such that $\epsilon_i \to 0$. Given ϵ_i, there exists δ_i such that every closed path in a δ_i-subset of U can be shrunk to a point in an ϵ_i-subset of U, since U is $1-ULC$. For ϵ_i, let T_{j_i} be the first member of the collection $\{T_i\}$ such that $j_i > j_{i-1}$ (if $i > 1$) and the diameter of the inverse image of the boundary of the star of each $v \in T_{j_i}$ is less than δ_i. We notice that the closed star

of $v \in T_{j_i}$ is 2-cell by Lemma 3.2, so it has a simple closed curve boundary, say $\partial\Delta_{i,v}$. Further, $Bd(M^3)$ is orientable [Mo, pg.170], so that it is a sphere with n handles.

Since S can be collared into U we can pull $f^{-1}(\partial\Delta_{i,v})$ into U along the collar, by a very small move. Let g be the homeomorphism giving the collar. Then the simple closed curve $g[f^{-1}(\partial\Delta_{i,v}) \times 1/2]$ can be shrunk to a point in an ϵ_i-subset of U.

Without loss of generality, we may assume that the singular disk $D'_{i,v}$ obtained by shrinking the simple closed curve $g[f^{-1}(\partial\Delta_{i,v}) \times 1/2]$ to a point, has no singular points on its boundary. In fact, suppose that there are singular points on the boundary. We notice that the singular disk $D'_{i,v}$, which is produced by the definition of $1-ULC$, is entirely in U. Therefore the distance between $f^{-1}(\partial\Delta_{i,v})$ and the singular disk $D'_{i,v}$ is nonzero, and there is an annulus between them in U. We tack onto $D'_{i,v}$ the collar beginning at $g[f^{-1}(\partial\Delta_{i,v}) \times 1/4]$, which is a simple closed curve lying in this annulus. Then the new singular disk

$$g[f^{-1}(\partial\Delta_{i,v}) \times [1/4, 1/2)] \cup D'_{i,v}$$

has no singular points on its boundary. Consequently we can consider the singular disk $D'_{i,v}$ as having no singular points on its boundary.

Hence, from the Topological Dehn's Lemma [Re], for every (small) neighborhood W of the singular set of $D'_{i,v}$, there exists a real topological disk $D_{i,v}$ in $D'_{i,v} \cup W$, such that $D_{i,v}$ differs from $D'_{i,v}$ (setwise), only inside W. Thus, we may assume that $diam(D_{i,v}) < 2\epsilon_i$, and that the collection $\{g[f^{-1}\partial\Delta_{i,v}) \times [0,1/2)] \cup D_{i,v}\}_{v\in K^0_{j_i}}$, is a collection of crosscaps to the boundary of U of diameter $< 3\epsilon_i$, induced by the triangulation T_{j_i}. Let us call this collection $\{Q_{i,v}\}_{v\in K^0_{j_i}}$. Then the collection $\{f(Q_{i,v})\}_{v\in K^0_{j_i}}$ is a finite collection of crosscaps to $Bd(M^3)$, obtained from the stars of the vertices of T_{j_i}. So the union of their corresponding domains and $Bd(M^3)$ forms a neighborhood of $Bd(M^3)$ in M^3, by Lemma 3.3. It follows that the union of the

corresponding domains of the crosscaps in the collection $\{Q_{i,v}\}_{v \in K^{o}_{j_i}}$ plus $Bd(U)$ is a neighborhood of $Bd(U)$ in $Cl(U)$.

Since ϵ_i can be chosen to be arbitrarily small, the conditions of Definition (7) are satisfied. Consequently, we have a prime end structure on U induced by f. □

<u>*Theorem A.3.8*</u> *If (U, f, M^3) is a bubble triple, then the map f is a C-transformation.*

Proof. We first show that the image of a crosscap is a crosscap. Since the map f is monotone on $Cl(U)$ and the closure of a crosscap is a disk, it follows that, for every crosscap Q, $Bd(f(Q))$ is a simple closed curve or a single point. It suffices to show that it cannot be degenerate. To this end, suppose that Q is a crosscap of U, with corresponding domain W, and such that $Bd(f(Q))$ is degenerate. Then any sufficiently small crosscap to the boundary of U, lying in W, maps to an open disk with degenerate boundary (tangent to $Bd(M^3)$). But, since U has a prime end structure induced by f (Theorem 3.7), we can find a crosscap $Q^* \subset W$ such that $f(Q^*)$ is a (small) crosscap of $Int(M^3)$ with $Bd(f(Q^*))$ being the boundary of a two-simplex of T_i, for some i. This contradicts our assumption that $Bd(f(Q))$ is degenerate. It follows that $f(Q)$ is a crosscap, for every crosscap Q in U.

We next show that f maps $Cl(Q)$ *homeomorphically* onto $Cl(f(Q))$. Suppose that f has singular points on $Bd(Q)$ for some Q in U, and let $p \in Bd(f(Q))$ such that $f^{-1}(p)$ is nondegenerate, necessarily an arc. Let S be the dense set of the definition of bubble domain. Then $p \notin f(S)$ since each point of S is an inverse set. Thus, we can take a chain of crosscaps $\{R_i\}$, where $Bd(R_i)$ is the boundary of the star of a vertex from the triangulation $\{T_i\}$ in the definition of bubble domain, and p lies in the interior of each associated two cell on $Bd(M^3)$. We also notice that $f^{-1}(p)$ is an arc $[a, b]$, since f is a monotone map. Since $Bd(R_i) \cap Bd(f(Q))$ has two points a_i, b_i for sufficiently large i, we consider the two sequences $\{a_i\}$ and $\{b_i\}$ which approach p. Then the two sequences $\{f^{-1}(a_i)\}, \{f^{-1}(b_i)\}$ lie on $Bd(Q)$ and

approach the opposite endpoints of the arc $f^{-1}(p)$, respectively. Thus, the points $f^{-1}(a_i)$ and $f^{-1}(b_i)$ are separated by some fixed positive number α. This contradicts the fact that $f^{-1}(T_i^1)$ forms a 1-dimensional ϵ_i-triangulation, where $\{\epsilon_i\}$ tends to 0, which fact comes from the definition of bubble domain. This contradiction shows that f maps $Cl(Q)$ homeomorphically onto $Cl(f(Q))$, and proves Condition (2) of the definition of C-transformation.

We now show that, for every chain of crosscaps $\{Q_i\}$, $\{f(Q_i)\}$ is a chain of crosscaps in M^3. Since f is uniformly continuous on $Cl(U)$, it suffices to show that $Bd(f(Q_i)) \cap Bd(f(Q_{i+1})) = \emptyset$. Suppose that there exists a chain of crosscaps $\{Q_i^*\}$ such that, for some i, $Bd(f(Q_i^*)) \cap Bd(f(Q_{i+1}^*)) \neq \emptyset$. Let $p \in Bd(f(Q_i^*)) \cap Bd(f(Q_{i+1}^*))$ and consider $f^{-1}(p)$. Then $f^{-1}(p)$ contains the points a_i and a_{i+1} in Q_i^* and Q_{i+1}^*, respectively, as well as a continuum containing both these points, since f is monotone. We again take a chain of crosscaps $\{R_i\}$ from the definition of bubble domain, such that the prime end induced by $\{R_i\}$ has p as its principal point, as in the above paragraph. Let $a_{i,j}$ and $a_{i+1,j}$ be points in $Bd(f(Q_i^*) \cap R_j$ and $Bd(f(Q_{i+1}^*) \cap R_j$ respectively, for $j = 1, 2, \ldots$. Then $\{f^{-1}(a_{i,j})\}$ and $\{f^{-1}(a_{i+1,j})\}$ converge to a_i and a_{i+1}, respectively, which are a positive distance apart. Therefore we have a contradiction similar to that in the above paragraph. This completes the proof of Condition (1) of the definition of $C-$transformation.

Since the collection $\{f^{-1}(T_i)\}$ is a collection of 1-dimensional ϵ_i-triangulations of $Bd(U)$, the set of images of small crosscaps and their complementary domains which are induced by f and $\{T_i\}$, clearly satisfies Condition (3) of the definition of $C-$transformation. This completes the proof of Theorem 3.8. □

Corollary A.3.3 (Induced Homeomorphism Theorem for Bubble Domains) *Let (U, f, M^3) be a bubble triple (that is, U is a bubble domain and f is its associated homeomorphism onto the interior of the compact 3-manifold-with-nonempty-boundary, M^3). If h is a homeomorphism of $Cl(U)$ onto itself, then the induced homeomorphism, fhf^{-1},*

of $Int(M^3)$ *onto itself can be extended to a homeomorphism* $\overline{fhf^{-1}}$ *of* M^3 *onto itself.* □

Proof. By the definition of bubble triple, U is an admissible domain and by Theorem 3.8, f is a C-transformation of U onto the interior of the compact 3-manifold, M^3. Thus, by Theorem 3.6, the Induced Homeomorphism holds. □

A.4 The Hilbert-Smith Conjecture

The following, known as the generalized Hilbert-Smith Conjecture, is the classic unsolved problem of topological transformation groups:

HILBERT-SMITH CONJECTURE : *If G is a compact group acting effectively on a manifold, then G is a Lie group.*

It is well known that this conjecture is equivalent to each of the following:

I. *A p-adic group cannot act effectively on a manifold*, and

II. *A compact 0-dimensional infinite group cannot act effectively on a manifold.*

The equivalence follows immediately from the following proposition, which was stated by Frank Raymond in [R1] as being a consequence of the structure theorems for abelian groups.

Proposition A.4.1 *If G is a compact non-Lie group acting effectively on a manifold, then G contains a p-adic group, for some prime p.*

A proof of Proposition 4.1 is not trivial and, as far as the authors know, has never appeared in the literature. Thus, in [L2], Lee provides a complete proof.

In this section, we will discuss the relationship between the Hilbert-Smith conjecture and three dimensional prime end theory. In fact, we apply our three dimensional prime end theory to obtain some results about a hypothesized p-adic action on E^3 or S^3. Our main results say that if there exists a p-adic action with an invariant bubble domain, then there exists a p-adic action with nonempty open set in its fixed point

set and with compact support. Thus there would exist p-adic actions supported on arbitrarily small open sets.

Before proceeding, we wish to acknowledge receipt of a recent preprint from Louis McAuley [Mc], asserting a proof of the H-S conjecture in full generality. We have not verified the results of [Mc], but we note that his preprint contains an excellent history of the problem, as well as a lengthy bibliography for the interested scholar.

A.4.1 Some Additional Equivalences

J.S. Lee has obtained several additional theorems related to the Hilbert-Smith Conjecture. In [L1], he proved that regularly almost periodic is equivalent to nearly periodic for homeomorphisms on compact metric spaces and gave an example to show that the above is false without the compactness assumption. He also proved that each of the following statements is equivalent to the Hilbert-Smith conjecture on a compact 3-manifold, M^3.

1. *If h is almost periodic on M^3, with h=identity on ∂M^3, then h=identity on M^3.*

2. *If h is regularly almost periodic on M^3, with h=identity on ∂M^3, then h=identity on M^3.*

3. *If h is regularly almost periodic on M^3, then h is periodic on M^3.*

4. *If h is nearly periodic on M^3, then h is periodic on M^3.*

5. (Newman's property on regularly almost periodic homeomorphisms). *Let h be a regularly almost periodic homeomorphism of M^3 onto itself. Then there exists $\epsilon > 0$ such that every h^i action on M^3, with $d(x, h^{iZ}(x)) < \epsilon$ for $i \in Z_+$, $x \in M^3$, is trivial.*

A.4.2 Invariant Open Sets

In this section, we use Bing's concepts of *Property S* and *ϵ-sequential growth* to construct a locally connected, invariant domain for a nontrivial compact subgroup of an arbitrary p-adic action on S^3 or E^3. The reader should consult [H-S] for a detailed discussion of Property S and ϵ-sequential growth, and proofs of their properties. Before beginning, however, we briefly review the definitions and properties we use.

Property S. Let K be a subset of a metric space X. Then K is said to have property S iff, for any $\epsilon > 0$, K can be expressed as the union of a finite number of connected sets, each of diameter less than ϵ.

ϵ-growth. Let H, K be subsets of a metric space X, and $\epsilon > 0$. Then H is said to ϵ-grow into K (or K is said to be an ϵ-growth of H) iff,

(i) $x \in K$ implies that there exists a connected subset M of X, such that $x \in M$, $M \cap H \neq \emptyset$, and $diam(M) < \epsilon$.

(ii) There exists $\alpha > 0$ such that, if M is any connected subset of X satisfying $diam(M) < \alpha$ and $M \cap H \neq \emptyset$, then $M \subset K$.

ϵ-sequential growth. Let H, K be subsets of a metric space X, and $\epsilon > 0$. Then K is said to be an ϵ-sequential growth of H (or H is said to grow ϵ-sequentially into K) iff, there is a sequence of positive numbers $\{\epsilon_i\}$ and a sequence of subsets $\{H_i\}$ of X such that

(i) H_1 is an ϵ_1-growth of H, and H_{j+1} is an ϵ_{j+1}-growth of H_j, for $j \in N$.

(ii) $K = \cup_{i \in N} H_i$.

(iii) $\sum_{i \in N} \epsilon_i \leq \epsilon$.

Proposition A.4.2 *[H-S, pg. 212]. Let K be a subset of a metric space X. If K has property S, then K is locally connected. If K is compact, then the converse is also true.* □

Proposition A.4.3 [H-S, pg. 215]. If K is a subset of a metric space X and K has property S, then $\overline{K}$ has property S. Thus, if K has property S, then $\overline{K}$ is locally connected. □

Proposition A.4.4 [H-S, pg. 216]. Let X be a metric space with property S, H and K subsets of X, and $\epsilon > 0$. If K is an ϵ-sequential growth of H, then K has property S and is open in X. □

We now prove the main theorem of this section. Originally, the proof comes from Brechner [B1,3].

Theorem A.4.1 Let A_p be a p-adic group acting on E^3 or S^3. Then, for each $\eta > 0$, there exist both a compact 0-dimensional subgroup, H, and a locally connected domain with compact, locally connected closure, $\overline{E^*}$, such that $\overline{E^*}$ is fully invariant under H.

Proof. Let g be the generator of the p-adic group A_p in the sense of P. A. Smith [S3]. Since g is nearly periodic, g is regularly almost periodic by [L2]. Let $\eta > 0$. Then there exists $\delta > 0$ such that $diam(g^n(\delta\text{-set})) < \eta/3$ for $n \in N$, since the set of powers of g is equicontinuous. Moreover, for this $\delta > 0$, there exists the positive integer, n_δ, such that $g^{n_\delta i}(x) \in B_\delta(x)$ for every $i \in Z$ and $x \in X$, since g is regularly almost periodic. Fix $x_0 \in E^3$, and let $D_1 = \cup_{i \in Z} (g^{n_\delta i}(B_\delta(x_0)))$. Then D_1 is invariant under g^{n_δ} and $diam(D_1) \leq \eta$.

Now we take $\epsilon > 0$ and a decreasing sequence of positive numbers $\{\epsilon_i\}$ such that $\sum \epsilon_i < \epsilon < \eta/2$. Given ϵ_1, there exists δ_1 such that $diam(g^{n_\delta i}(\delta_1\text{-set})) < \frac{\epsilon_1}{3}$ for every $i \in Z$. We consider the open cover $\{U_{1,1}, U_{1,2}, ..., U_{1,k_1}\}$ of $\overline{D_1}$ with mesh less than δ_1. Then $diam(g^{n_\delta i}(U_{1,j})) < \frac{\epsilon_1}{3}$, for $j = 1, ..., k_1$. Let

$$D_2 = \cup_{i \in Z}(g^{n_\delta i}(\cup_{i=1}^{k_1} U_{1,i})).$$

Then D_2 is invariant under g^{n_δ}. We show that D_2 is an ϵ_1-growth of D_1. We prove parts (i) and (ii) of the definition of ϵ-growth.

Proof of (i). Let $x \in D_2 - D_1$. Then there exists k such that $x \in g^{n_i k}(U_{1,j})$ for some j. But $g^{n_i k}(U_{1,j})$ is connected, meets D_1, and has diameter less than ϵ_1. So (i) holds.

Proof of (ii). $\overline{D_1}$ and $S^3 - D_2$ are disjoint compact subsets of S^3 and thus a positive distance apart, say $2\alpha_2$. Then, the α_2 neighborhood of $\overline{D_1}$, and therefore the α_2 neighborhood of D_1, is a subset of of D_2.

It is clear that we may continue the process inductively, obtaining at the ith stage, a connected open set D_i which is an ϵ_{i-1}-growth of D_{i-1}. Let $E^* = \cup_{i=1}^{\infty} D_i$. Then by Proposition 4.4, E^* is open, has property S, and has diameter less than $2\epsilon < \eta$. Thus $\overline{E^*}$ is a locally connected continuum by Proposition 4.3. Further $\overline{E^*}$ is invariant under g^{n_i}, so it is invariant under the action of the compact 0-dimensional group which is generated by g^{n_i} in the sense of P. A. Smith [S3]. □

A.4.3 Induced Actions on Manifolds

Lemma A.4.1 *Let U be a bubble domain and let g be a nearly periodic transformation acting on $Cl(U)$. Then the induced homeomorphism, fgf^{-1}, is a nearly periodic transformation acting on $Int(M^3)$.*

Proof. Let $\{\mathcal{U}_i\}$ be a complete system which is invariant and such that the mesh has limit 0. For each i, we consider

$$\{f(U_{i,1}), f(U_{i,2}), ... f(U_{i,p_i})\}.$$

Then $(fgf^{-1})^{n_i}(f(U_{i,j})) = fg^{n_i}(U_{i,j}) = f(U_{i,j})$ for $j = 1, ..., p_i$, where n_i is the period of $\mathcal{U}_i$. So $f(\mathcal{U}_i)$ forms an invariant open cover of $Int(M^3)$, and the sequence $\{mesh\,(f(\mathcal{U}_i))\}$ has limit 0, since the C-transformation f is the restriction of the uniformly continuous map $f : Cl(U) \to M^3$. It follows that fgf^{-1} is nearly periodic on $Int(M^3)$. □

Lemma A.4.2 *Let g be a nearly periodic transformation acting on $Int(M^3)$. If g can be extended to a homeomorphism $\overline{g}$ of M^3 onto itself, then $\overline{g}$ is nearly periodic.*

Proof. Recall that, since g is nearly periodic on $Int(M^3)$, it is regularly almost periodic on $Int(M^3)$. By [L2], regularly almost periodic and nearly periodic are equivalent on *compact* spaces. Thus, it is sufficient to show that $\overline{g}$ is regularly almost periodic on all of M^3.

To this end, let $\epsilon > 0$. We must find an integer $n_* > 0$ such that $dist(x, \overline{g}^{kn_*}(x)) < \epsilon$, for all $x \in M^3$, and all integers k. Take $\epsilon/3$. Then there exists $n' > 0$ such that $dist(x, g^{kn'}(x)) < \epsilon/3$ for all $x \in Int(M^3)$. Let $x_n \to x$, where $x_n \in Int(M^3)$. Then $\overline{g}^{kn'}(x_n) \to \overline{g}^{kn'}(x)$. Then there exists $i > 0$ such that for all $j > i$, $dist(x_j, x) < \epsilon/4$ and $dist(\overline{g}^{kn'}(x_j), \overline{g}^{kn'}(x)) < \epsilon/4$. By the triangle inequality, it follows that $dist(x, \overline{g}^{kn'}(x)) < \epsilon$. We take n_* to be n'. This completes the proof. □

<u>Theorem A.4.2</u> *Let g be a nearly periodic transformation acting on the closure of a bubble domain, $Cl(U)$. Then there exists a positive integer n such that g^n acts as the identity on $Bd(U)$.*

Proof. Let g be nearly periodic on $Cl(U)$. Then the homeomorphism (fgf^{-1}) of $Int(M^3)$ onto itself can be extended to a homeomorphism $\overline{g}$ of M^3 onto itself which is nearly periodic, by Lemmas 4.1 and 4.2. Note that on a 2-manifold, a nearly periodic homeomorphism is periodic [L2]. Therefore there exists a positive integer n such that $\overline{g}^n$ acts as the identity on $Bd(M^3)$.

Also, from the definition of bubble domain, $Bd(U)$ contains a dense subset S which, in turn, contains a sequence of 1-dimensional ϵ_i-triangulations, where $\epsilon_i \to 0$. Further, since the map f induces a prime end structure on U coming from these 1-dimensional ϵ_i-triangulations (by Theorem 3.7), we see that the union of these 1-dimensional ϵ_i-triangulations, say T, is dense in $Bd(U)$, and each point of T is the principal point of some chain of crosscaps of U induced by M and f.

We claim that each point of T is fixed under g^n. In fact, let $p \in T$, and let p be the principal point of the prime end $E = \{Q_i\}$. Then $f(p) = p'$ is the principal

point for the prime end E' of $Int(M^3)$, where $E' = \{f(Q_i)\}$. Now, since p' is fixed under $\overline{g}^n$, $\overline{g}^n(E') = E'$. That is, $\{g^n(f(Q_i))\}$ is equivalent to $\{f(Q_i)\}$, so that these prime ends are the same. Thus, their inverses under f are the same. This means that $f^{-1}(fgf^{-1})^n(E') = f^{-1}(E')$, which means that $f^{-1}(fg^nf^{-1}(E')) = f^{-1}(E')$, or that $g^nf^{-1}(E') = f^{-1}(E')$. But $f^{-1}(E') = E$, so that we have $g^n(E) = E$. Since the prime ends $g^n(E)$ and E are the same, and they each have a single principal point, they have the *same* principal point, p. Therefore, $g^n(p) = p$. Since each point of T is fixed under g^n, and T is dense in $Bd(U)$, the theorem follows. □

<u>Theorem A.4.3</u> *Let A_p be a p-adic group acting effectively on $S^3(E^3)$, and let U be a bounded domain whose closure is invariant under A_p. If U is a bubble domain, then there exists a compact 0-dimensional group acting on $S^3(E^3)$ which is the identity on an open set, and thus there exist compact 0-dimensional groups which are supported on arbitrarily small open sets.*

Proof. Let A_p be a p-adic group acting on S^3, and let g be the generator of A_p in the sense of P. A. Smith [S3]. Then $g^n = id.$ on $Bd(U)$ by Theorem 4.2. We consider the compact, 0-dimensional, infinite subgroup G of A_p, which is generated by g^n. Since G is an effective action, it must be an effective action on at least one of the complementary domains, say V, of $Bd(U)$. Since $g^n = id.$ on $Bd(U)$, we can extend the action of G on $Cl(V)$ to an action G' on S^3, so that G' acts on V as G does, while G' acts as the identity on $S^3 - V$.

Since the complement of $Cl(V)$ in S^3 contains an open set, $Cl(V)$ can be conjugated into arbitrarily small open subsets of S^3 by homeomorphisms of S^3 onto itself. We conclude that there exists a compact 0-dimensional group which can act on S^3 in such a way that it is supported on arbitrarily small open sets.

If we start with E^3 instead of S^3, we can think of the action as an action on S^3 by adding the point at infinity, and making that point a fixed point of each element

of A_p. Then the above argument applies, with infinity as part of the open set of support. If one removes from S^3 any point which is in the open fixed point set, then the result is an action on E^3, with the set of support being compact. Thus it can be conjugated into an arbitrarily small open subset of E^3. □

It would be nice to extend the above argument to an arbitrary admissible triple, (U, ϕ, M^3), where U is a bounded, connected, $1-ULC$ domain in E^3 which is homeomorphic to the interior of some compact 3-manifold, M^3, and ϕ is a C-transformation onto the interior of M^3. That is, we ask whether the results of Section 4.3 hold - Does a nearly periodic homeomorphism on $Cl(U)$ induce a nearly periodic homeomorphism on $Int(M^3)$? - when U is only an *admissible* domain, but not a *bubble* domain.

The following example shows that the above *argument* cannot be used in general for admissible triples. Let X be the compact unit cube in E^3, and let D be a closed 2-cell in X, with an arc of its boundary in the boundary of X. Let U be the open set $Int(X) - D$, and let W be the small open (disconnected) subset of U which is the interior of the infinitely spiraled horn, as indicated in the diagram below. Under the C-transformation ϕ, the set W maps to two infinite sequences with two limit points, as indicated in the diagram. If W were a small member of a finite invariant collection of open sets, as in Lemma 4.1, its image would not be small. Thus, the argument of that lemma does not carry over.

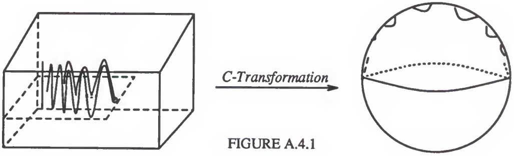


FIGURE A.4.1

We do not know whether the theorem remains true in general. Nevertheless, under some more general conditions, a nearly periodic transformation *is* preserved by a C-transformation.

Lemma A.4.3 *Let (U, ϕ, M^3) be an admissible triple, and let g be a nearly periodic transformation acting on $Cl(U)$. Suppose there exists a complete system $\{\mathcal{U}_i\}$ of $\mathcal{U}$ such that, for each $U_{i,j}$, the number of components of $U_{i,j}$ is finite. Then the induced homeomorphism, $\phi g \phi^{-1}$, is nearly periodic on $Int(M^3)$.*

Proof. Let $\{\mathcal{U}_i\}$ be a complete invariant system such that the mesh has limit 0. For each i, we consider

$$\{\phi(U_{i,1}), \phi(U_{i,2}), ..., \phi(U_{i,p_i})\}.$$

Then $(\phi g \phi^{-1})^{n_i}(\phi(U_{i,j}) = \phi g^{n_i}(U_{i,j}) = \phi(U_{i,j})$ for $j = 1, ..., p_i$, where n_i is the period of $\mathcal{U}_i$. So $\phi(\mathcal{U}_i)$ forms an invariant open cover of $Int(M^3)$. In general, $mesh(\phi(\mathcal{U}_i))$ does not have limit 0. So we consider the components of $U_{i,j}$. If the number of components of each $U_{i,j}$ is finite, then we split each $U_{i,j}$ into its components, $\{U_{i,j_1}, U_{i,j_2}, ..., U_{i,j_{n_{i,j}}}\}$, obtaining from $\mathcal{U}_i$, a new open cover, $\mathcal{V}_i$. Then $\mathcal{V}_i$ is invariant under the action of g, since, for every $j \in j_1, j_2, ..., j_{n_{i,j}}$, $g^{k_j}(U_{i,j}) = U_{i,j}$ for some k_j; and the period of $\mathcal{V}_i$ is $n_k \Pi(k_j)$. Note that each element of $\mathcal{V}_i$ is connected and that $mesh(\mathcal{V}_i) \to 0$. Now, by Theorem 3.2, ϕ induces a prime end structure on U. Therefore, small components of $\mathcal{V}_i$ near $Bd(U)$ are contained in the corresponding domains of small crosscaps. Since a C-transformation is uniformly continuous on each crosscap, the images of such crosscaps are small. But, on a manifold, the corresponding domain of a small crosscap is small. It follows that $\phi(\mathcal{V}_i)$ has small mesh. Since $\phi(\mathcal{V}_i)$ is invariant under $\phi g \phi^{-1}$, this homeomorphism is nearly periodic on $Int(M^3)$. □

Lemma A.4.4 *Let (U, ϕ, M^3) be an admissible triple in E^3, and let A be the set of points of $Bd(U)$ which are accessible through a chain of crosscaps of U. Then*

(1) A is dense in $Bd(U)$

(2) Each point of A determines a prime end of U, and

(3) The set of prime ends of $Bd(M^3)$ which correspond to the prime ends of U determined by A, forms a dense subset of $Bd(M^3)$.

<u>Theorem A.4.4</u> Let g be a nearly periodic transformation acting on an admissible domain $Cl(U)$, and let (U, ϕ, M^3) be an admissible triple satisfying the hypotheses of Lemma 4.3. Then there exists n such that g^n acts as the identity on $Bd(U)$.

Proof. Suppose g acts on $Cl(U)$. Then the homeomorphism $(\phi g \phi^{-1})$ of $Int(M^3)$ onto itself can be extended uniquely to a nearly periodic homeomorphism $\overline{\phi g \phi^{-1}}$ of M^3 onto itself, by Lemma 2.3. Note that a nearly periodic homeomorphism on a 2-manifold is periodic. Therefore there exists n such that $(\overline{\phi g \phi^{-1}})^n$ acts as the identity on $Bd(M^3)$.

By Lemma 4.4, the set of points, A, of $Bd(U)$ that are accessible through a chain of crosscaps of U, forms a dense subset of $Bd(U)$, and each such point defines a prime end of U. By Theorem 3.2, the prime end structure of $Bd(U)$ is induced by the prime end structure of $Int(M^3)$. It follows that A is pointwise fixed under g^n, and thus, $Bd(U)$ is fixed under g^n. In fact, let p be an accessible point in A. Then $g^n(p) = g^n(\phi^{-1}(p'))$ for some p' in $Bd(M^3)$. But $\phi g^n \phi^{-1}(p') = p'$, so that $g^n \phi^{-1}(p') = \phi^{-1}(p') = p$. Therefore $g^n(p) = p$. □

<u>Theorem A.4.5</u> Let A_p be a p-adic group acting effectively on S^3, and let U be a bounded domain whose closure is invariant under A_p. If (U, ϕ, M^3) is an admissible triple satisfying Lemma 4.3, then there exists a compact 0-dimensional group acting on S^3 in such a way that it is the identity on an open set. Thus there exist compact 0-dimensional groups which are supported on arbitrarily small open sets.

Proof. Let A_p be a p-adic group acting effectively on S^3, and let g be the generator of A_p in the sense of P. A. Smith [S3]. Then $g^n = id$ on boundary of $Cl(U)$

for some n, by Theorem 4.4. We consider the compact 0-dimensional infinite group G, which is generated by g^n. Since $g^n = id$ on $Bd(U)$, we can extend the action of G on U to an action on S^3 in such a way that G acts on $S^3 - U$ as the identity.

Therefore we conclude that there exists a compact 0-dimensional group acting on S^3 which is the identity on an open set. (A conjugate of) this group action can be made to be supported on arbitrarily small open sets by conjugating $Cl(U)$ into arbitrarily small open subsets of S^3. □

A.4.4 Framework for a Possible Proof of the H-S Conjecture

It is conceivable that prime end theory could be used to prove the Hilbert-Smith Conjecture, if it is indeed true. More specifically, suppose there is a p-adic action, A_p, on S^3. If one could construct a bubble domain, U, which remains invariant under this action, then an action by a homomorphic image of A_p is induced on the prime end compactification - a compact 3-manifold, M^3, with 2-manifold boundary. The induced action must be finite on $Bd(M^3)$, making the original action finite on the boundary of the bubble domain. Thus, if the bubble domain could be constructed so that an arbitrarily chosen point $p \in E^3$ is a point of its boundary, then p is a periodic point of every element of the action. Since p was arbitrary, each homeomorphism of the action is pointwise periodic. By Montgomery [Mtg], a pointwise periodic homeomorphism of a manifold onto itself must be periodic, so that every element of A_p is periodic. However, no element of a p-adic group can be periodic. This contradiction would establish the Hilbert-Smith Conjecture.

Remark. A complete proof of the Hilbert-Smith Conjecture *for* E^2 along these lines was presented in 1984 by Brechner in the unpublished manuscript, [Br3].

A.5 Open Problems

The results of this paper lead to the following interesting and important open questions.

1. Let G be a p-adic action on $E^3(S^3)$. Does there necessarily exist an invariant *bubble* domain? If so, can we build such a domain with an arbitrarily designated point $p \in E^3(S^3)$ as a point of the boundary? as an accessible point of the boundary?

2. Let G be a p-adic action on $E^3(S^3)$. Does there necessarily exist an invariant *admissible* domain?

3. Characterize the continua, including the closures of bounded domains, that can remain invariant under p-adic actions.

4. Let U be a bounded, simply connected, $1-ULC$ domain in E^3. By work of L. Husch [Hu2], C.H. Edwards [Ed], and C. T. C. Wall [Wa], we know that U is homeomorphic to the interior of the unit 3-ball, B^3. Does there necessarily exist a C-transformation $f: U \to Int(B^3)$? That is, is every bounded, simply connected, $1-ULC$ domain in E^3 admissible? Also by work of L. Husch [Hu2], if we omit the simply connected hypothesis and add some other conditions, U is homeomorphic to the interior of a compact 3-manifold. Therefore we have the same question as above: Is such a domain admissible?

5. In general, let U be a bounded, connected, $1-ULC$ domain in E^3 which is homeomorphic to the interior of a compact 3-manifold, M^3, with nonempty boundary. Must U be admissible? That is, does there exist a C-transformation of U onto $Int(M^3)$?

6. Characterize the admissible domains and/or continua in E^3.

7. Characterize those domains which admit C-transformations.

8. If U is a domain which admits a C-transformation, is U necessarily $1-ULC$? That is, is the $1-ULC$ hypothesis necessary in the definition of admissible domain?

9. Characterize those domains which have a prime end structure.

10. Does the Induced Homeomorphism Theorem hold for arbitrary admissible domains?

11. If g is nearly periodic on an *admissible* domain U, is $\phi g \phi^{-1}$ nearly periodic on $Int(M^3)$?

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